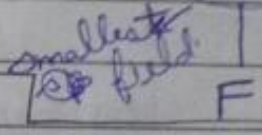

Results- Let F be a field and let $f(x) \in F[x]$ be irreducible over F . If α is a zero of $f(x)$ in some extension E of F , then $F(\alpha)$ is isomorphic to $\frac{F[x]}{\langle f(x) \rangle}$. $F(\alpha)$

Concepts- If $\deg f(x) = n$ then $[F(\alpha) : F] = n$



and $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ form a basis of the vector space $F(\alpha)$ over F .

Example ① Consider the polynomial $f(x) = x^6 - 2 \in \mathbb{Q}[x]$. find $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}]$.

Example ② Consider the polynomial $x^3 - 2 \in \mathbb{Q}[x]$. find $[\mathbb{Q}(2^{1/3}) : \mathbb{Q}]$.

Solution (Ex ②)

$f(x) = x^3 - 2 \in \mathbb{Q}[x]$

by Eisenstein's criteria, $f(x)$ is irreducible over \mathbb{Q} .

$x^3 - 2 = 0 \Rightarrow x^3 = 2 \Rightarrow x = 2^{1/3}$ is real root

take $\alpha = 2^{1/3}$

$\therefore F(\alpha) = \mathbb{Q}[2^{1/3}]$ is smallest field which is extension of \mathbb{Q} .

$\mathbb{Q}[2^{1/3}]$

and $\mathbb{Q}[2^{1/3}] \cong \frac{\mathbb{Q}[x]}{\langle x^3 - 2 \rangle}$

degree of extension of $\mathbb{Q}(2^{1/3})$ over $\mathbb{Q} = [\mathbb{Q}[2^{1/3}] : \mathbb{Q}] = 3$

$$\mathbb{Q}(2^{1/3}) \cong \frac{\mathbb{Q}[x]}{\langle x^3 - 2 \rangle}$$

$$x^3 - 2 = (x - 2^{1/3})(x - \alpha)(x - \bar{\alpha})$$

basis of $\mathbb{Q}(2^{1/3})/\mathbb{Q} = \{a^0, a^1, a^2\}$
 $= \{2^{1/3}{}^0, 2^{1/3}{}^1, 2^{1/3}{}^2\}$
 $= \{1, 2^{1/3}, 2^{2/3}\}$

Example 1

$f(x) = x^6 - 2 \in \mathbb{Q}[x]$ is irreducible over \mathbb{Q}
 by Eisenstein criteria of irreducibility

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{F}[x]$$

if \exists prime $p \nmid n$:

$$p \nmid a_0, p \nmid a_1, \dots, p \nmid a_{n-1}, \text{ but } p \nmid a_n$$

$$\text{and } p^2 \nmid a_0$$

then $f(x)$ is irreducible over \mathbb{F} .

Choose $p = 2$, $p \nmid 6$, $p \nmid 1$

$$p^2 \nmid (-2)$$

$f(x) = x^6 - 2$ is irreducible over $\mathbb{Q}(x)$

$$\therefore x^6 - 2 = 0 \Rightarrow x = 2^{1/6} = \alpha$$

$$\mathbb{Q}(2^{1/6})$$

$$\mathbb{Q}$$

$$[\mathbb{Q}(2^{1/6}) : \mathbb{Q}] = 6 \quad \underline{\text{Ans}}$$

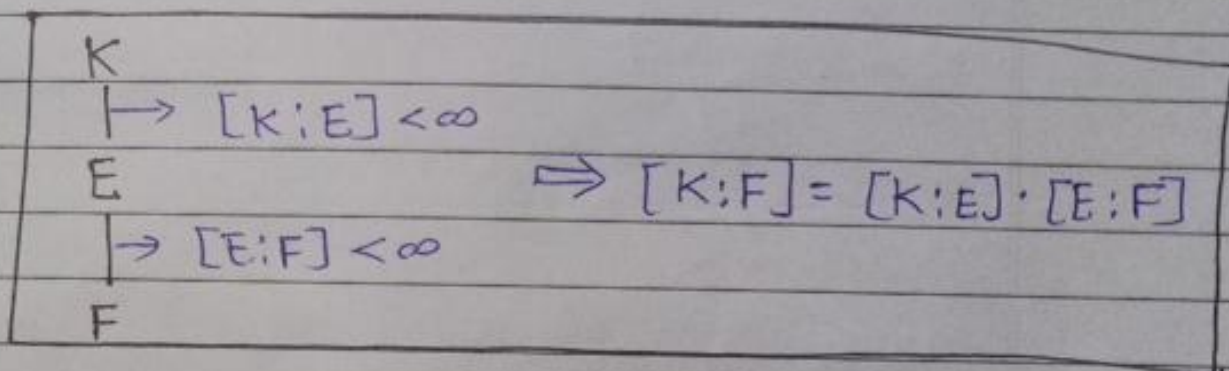
basis of $\mathbb{Q}(2^{1/6})/\mathbb{Q} = \{a^0, a^1, a^2, a^3, a^4, a^5\}$

$$= \{1, 2^{1/6}, 2^{2/6}, 2^{3/6}, 2^{4/6}, 2^{5/6}\}$$

and $\mathbb{Q}(2^{1/6}) \cong \frac{\mathbb{Q}[x]}{\langle x^6 - 2 \rangle}$

RESULT Let K be a finite extension of the field E , and E be a finite extension of field F then K is a finite extension of field F and

$$[K:F] = [K:E] \cdot [E:F]$$



* Ques find $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}]$

Sol:-

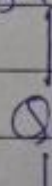
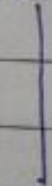
$$\mathbb{Q}[\sqrt{2}, i] = \mathbb{Q}(\sqrt{2})(i) = \{x + iy; x, y \in \mathbb{Q}(\sqrt{2})\}$$

$$\text{Basis} = \{1, i\}$$

$$\mathbb{Q}[\sqrt{2}, i]$$

$$\mathbb{Q}(\sqrt{2}, i)$$

$$[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] = 2$$



$$\mathbb{Q}$$

$$\mathbb{Q}$$

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2}, a, b \in \mathbb{Q}\}$$

$$\text{basis } \{1, \sqrt{2}\}$$

$$\dim [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$$

$$\text{or } [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$$

$$\therefore [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] \cdot [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$$

$$= 2 \cdot 2$$

$$= 4$$