Result:- Let F be a field and let $f(x) \in F[x]$ be irreducible over F. If a is a zero of $f(x)$ in some extension E of F, then $F(a)$ is isomorphic to $\frac{F[x]}{\langle f(x) \rangle}$.

$F(a)$ — smallest field
$|$
F

Concept:- If $\deg f(x) = n$
then $[F[a] : F] = n$
and
$\{a^0, a^1, a^2, \dots, a^{n-1}\}$ form a bases of the vector space $F(a)$ over F.

Example ① Consider the polynomial $f(x) = x^6 - 2 \in Q[x]$. find $[Q(\sqrt[6]{2}) : Q]$.

Example ② Consider the polynomial $x^3 - 2 \in Q[x]$. find $[Q(2^{1/3}) : Q]$.

Solution (Ex ②)

$f(x) = x^3 - 2 \in Q[x]$

by Einstien's criteria; $f(x)$ is irreducible over Q.

$x^3 - 2 = 0 \Rightarrow x^3 = 2 \Rightarrow x = 2^{1/3}$ is real root

take $a = 2^{1/3}$

$\therefore F[a] = Q[2^{1/3}]$ is smallest field which is extension of Q.

$Q[2^{1/3}]$
$|$
Q

and $Q[2^{1/3}] \cong \frac{Q[x]}{\langle x^3 - 2 \rangle}$

degree of extension of $Q(2^{1/3})$ over $Q = [Q[2^{1/3}] : Q] = 3$

$$\mathbb{Q}(2^{1/3}) \cong \frac{\mathbb{Q}[x]}{\langle x^3-2\rangle}$$

$$x^3-2 = (x-2^{1/3})(x-\alpha)(x-\bar{\alpha})$$

$$\text{basis of } \mathbb{Q}(2^{1/3})(\mathbb{Q}) = \{a^0, a^1, a^2\}$$
$$= \{(2^{1/3})^0, (2^{1/3})^1, (2^{1/3})^2\}$$
$$= \{1, 2^{1/3}, 2^{2/3}\}$$

<u>Example(2)</u>

$f(x) = x^6 - 2 \in \mathbb{Q}[x]$ irreducible over $\mathbb{Q}$

by Einstien criteria of irreducibility

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in F[x]$$

If $\exists$ prime p s.t.

$p/a_0, p/a_1, \dots, p/a_{n-1}$, but $p \nmid a_n$

and $p^2 \nmid a_0$

then $f(x)$ is irreducible over F.

Choose $p = 2$, $p/(2)$, $p \nmid 1$

$p^2 \nmid (-2)$

$f(x) = x^6 - 2$ is irreducible over $\mathbb{Q}[x]$

$\therefore$ $x^6 - 2 = 0 \Rightarrow x = 2^{1/6} = a$

$$\begin{array}{c} \mathbb{Q}(2^{1/6}) \\ | \\ \mathbb{Q} \end{array}$$

$[\mathbb{Q}(2^{1/6}) : \mathbb{Q}] = 6$ Ans

basis of $\mathbb{Q}(2^{1/6})(\mathbb{Q}) = \{a^0, a^1, a^2, a^3, a^4, a^5\}$

$$= \{1, 2^{1/6}, 2^{2/6}, 2^{3/6}, 2^{4/6}, 2^{5/6}\}$$

and $\mathbb{Q}(2^{1/6}) \cong \dfrac{\mathbb{Q}[x]}{\langle x^6-2\rangle}$

RESULT Let K be a finite extension of the field E and E be a finite extension of Field F then K is a finite extension of field F and

$$[K:F] = [K:E]\cdot[E:F]$$

$$\boxed{\begin{array}{l} K \\ \big| \rightarrow [K:E] < \infty \\ E \qquad\qquad\qquad\qquad \Rightarrow [K:F] = [K:E]\cdot[E:F] \\ \big| \rightarrow [E:F] < \infty \\ F \end{array}}$$

Ques find $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}]$

Sol^n:- $\mathbb{Q}[\sqrt{2}, i] = \mathbb{Q}(\sqrt{2})(i) = \{x+iy \,;\, x, y \in \mathbb{Q}[\sqrt{2}]\}$

Basis $= \{1, i\}$

$[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}[\sqrt{2}]] = 2$

$$\begin{array}{c} \mathbb{Q}[\sqrt{2}, i] \\ \big| \\ \big| \\ \mathbb{Q} \end{array} \qquad\qquad \begin{array}{c} \mathbb{Q}(\sqrt{2}, i) \\ \big| \\ \mathbb{Q}[\sqrt{2}] \\ \big| \\ \mathbb{Q} \end{array}$$

$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2},\ a, b \in \mathbb{Q}\}$

basis $\{1, \sqrt{2}\}$

$\dim[\mathbb{Q}[\sqrt{2}](\mathbb{Q})] = 2$

or $[\mathbb{Q}[\sqrt{2}] : \mathbb{Q}] = 2$

$$\therefore\ [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}[\sqrt{2}]]\cdot[\mathbb{Q}[\sqrt{2}] : \mathbb{Q}]$$
$$= 2\cdot 2$$
$$= 4$$