## Definition

The transportation problem is a special type of linear programming problem, where the objective is to minimize the cost of distributing a product from a number of sources to a number of destinations.

**The general mathematical model may be given as follows**

If $x_{ij}$ ($\geq 0$) is the number of units shipped from ith source to jth destination, then equivalent LPP model will be

Minimize $Z = \sum_{i=1}^{m} \sum_{j}^{n} c_{ij} x_{ij}$

Subject to

$\sum_{j=1}^{n} x_{ij} \leq a_i$ For i = 1,2, ....., m (supply)

$\sum_{i=1}^{m} x_{ij} \leq b_j$ For j = 1,2, ....., n (demand)

$x_{ij} \geq 0$ .

For a feasible solution to exist, it is necessary that total capacity equals total to the requirements. If $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$ i.e. If total supply = total demand then it is a balanced transportation problem otherwise it is called unbalanced Transportation problem. There will be (m + n - 1) basic independent variables out of (m x n) variables

**What are the understanding assumptions?**

1. Only a single type of commodity is being shipped from an origin to a destination.
2. Total supply is equal to the total demand.
   $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ , $a_i$ (supply) and $b_j$ (demand) are all positive integers.
3. The unit transportation cost of the item from all sources to destinations is certainly and preciously known.
4. The objective is to minimize the total cost.

**North West Corner Rule**

**Example 1:** The ICARE Company has three plants located throughout a state with production capacity 50, 75 and 25 gallons. Each day the firm must furnish its four retail shops $R_1$, $R_2$, $R_3$, & $R_4$ with at least 20, 20 , 50, and 60 gallons respectively. The transportation costs (in Rs.) are given below.

| Company | Retail | | | | Supply |
|---|---|---|---|---|---|
| | R1 | R2 | R3 | R4 | |
| P1 | 3 | 5 | 7 | 6 | 50 |
| P2 | 2 | 5 | 8 | 2 | 75 |
| P3 | 3 | 6 | 9 | 2 | 25 |
| Demand | 20 | 20 | 50 | 60 | |

The economic problem is to distribute the available product to different retail shops in such a way so that the total transportation cost is minimum?

**Solution:** Starting from the North West corner, we allocate min (50, 20) to $P_1R_1$, i.e., 20 units to cell $P_1R_1$. The demand for the first column is satisfied. The allocation is shown in the following table.

**Table 1**

| Company | Retail | | | | Supply |
|---|---|---|---|---|---|
| | R1 | R2 | R3 | R4 | |
| P1 | 3 (20) | 5 (20) | 7 (10) | 6 | 50 |
| P2 | 2 | 5 | 8 (40) | 2 (35) | 75 |
| P3 | 3 | 6 | 9 | 2 (25) | 25 |
| Demand | 20 | 20 | 50 | 60 | |

Now we move horizontally to the second column in the first row and allocate 20 units to cell $P_1R_2$. The demand for the second column is also satisfied.

Proceeding in this way, we observe that $P_1R_3 = 10$, $P_2R_3 = 40$, $P_2R_4 = 35$, $P_3R_4 = 25$. The resulting feasible solution is shown in the following table.

Here, number of retail shops (n) = 4, and
Number of plants (m) = 3.Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6.

**Initial basic feasible solution**

The initial basic feasible solution is $x_{11}=20$, $x_{12}=5$, $x_{13}=20$, $x_{23}=40$, $x_{24}=35$, $x_{34}=25$ and minimum cost of transportation=20 X 3 + 20 X 5 + 10 X 7 + 40 X 8 + 35 X 2 + 25 X 2 = 670

**Matrix Minimum Method**

**Example 2:** Consider the transportation problem presented in the following table**:**

| Factory | Retail shop | | | | Supply |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | |
| 1 | 3 | 5 | 7 | 6 | 50 |
| 2 | 2 | 5 | 8 | 2 | 75 |
| 3 | 3 | 6 | 9 | 2 | 25 |
| Demand | 20 | 20 | 50 | 60 | |

**Solution:**

| Factory | Retail shop | | | | Supply |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | |
| 1 | 3 | 5 20 | 7 30 | 6 | 50 |
| 2 | 2 20 | 5 | 8 | 2 55 | 75 |
| 3 | 3 | 6 | 9 20 | 2 5 | 25 |
| Demand | 20 | 20 | 50 | 60 | |

Number of basic variables = m + n –1 = 3 + 4 – 1 = 6.

**Initial basic feasible solution**

The initial basic feasible solution is $x_{12}$=20, $x_{13}$=30, $x_{21}$=20, $x_{24}$=55, $x_{233}$=20, $x_{34}$=5 and minimum cost of transportation=20 X 2 + 20 X 5 + 30 X 7 + 55 X 2 + 20 X 9 + 5 X 2 = 650.

## Vogel Approximation Method (VAM)

The Vogel approximation (Unit penalty) method is an iterative procedure for computing a basic feasible solution of a transportation problem. This method is preferred over the two methods discussed in the previous sections, because the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

**Example 3:** Obtain an Initial BFS to the following Transportation problem using VAM method?

| Origin | Destination 1 | 2 | 3 | 4 | Supply |
|---|---|---|---|---|---|
| 1 | 20 | 22 | 17 | 4 | 120 |
| 2 | 24 | 37 | 9 | 7 | 70 |
| 3 | 32 | 37 | 20 | 15 | 50 |
| Demand | 60 | 40 | 30 | 110 | 240 |

**Solution:** Since $\sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j$ , the given problem is balanced TP., Therefore there exists a feasible solution.

**Step -1**: Select the lowest and next to lowest cost for each row and each column, then the difference between them for each row and column displayed them with in first bracket against respective rows and columns. Here all the differences have been shown within first compartment. Maximum difference is 15 which is occurs at the second column. Allocate min (40,120) in the minimum cost cell (1,2).

**Step -2**: Appling the same techniques we obtained the initial BFS. Where all capacities and demand have been exhausted

**Table Initial basic feasible solution**

| | Destination | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Origin** | **1** | **2** | **3** | **4** | **Supply** | **Penalty** | | | | | |
| **1** | 20 | **22** | 17 | **4** | ~~123~~ | 13 | 13 | - | - | - | - |
| **2** | **24** | 37 | **9** | **7** | ~~70~~ | 2 | 2 | 2 | 17 | 24 | 24 |
| **3** | **32** | 37 | 20 | 15 | ~~50~~ | 5 | 5 | 5 | 17 | 32 | - |
| **Demand** | ~~60~~ | ~~40~~ | ~~30~~ | ~~110~~ | 240 | | | | | | |
| **Penalty** | 4 | 15 | 8 | 3 | | | | | | | |
| | 4 | - | 8 | 3 | | | | | | | |
| | 8 | - | 11 | 8 | | | | | | | |
| | 8 | - | - | 8 | | | | | | | |
| | 8 | - | - | - | | | | | | | |
| | 24 | - | - | - | | | | | | | |

The initial basic feasible solution is $x_{12}$=40, $x_{14}$=40, $x_{21}$=10, $x_{23}$=30, $x_{24}$=30, $x_{31}$=50.and minimum cost of transportation=22 X 40 + 4 X 80 + 24 X 10 + 9 X 30 + 7 X 30 + 32 X 50 = 3520.

**Optimality Test for Transportation problem**

There are basically two methods

a) Modified Distribution Method (MODI)
b) Stepping Stone Method.

**Modified Distribution Method (MODI)**

The modified distribution method, also known as MODI method or (u - v) method provides a minimum cost solution to the transportation problem. In the stepping stone method, we have to draw as many closed paths as equal to the unoccupied cells for their evaluation. To the contrary, in MODI method, only closed path for the unoccupied cell with highest opportunity cost is drawn.

**Steps**

1. Determine an initial basic feasible solution using any one of the three methods given below:

a) North West Corner Rule
b) Matrix Minimum Method
c) Vogel Approximation Method

2. Determine the values of dual variables, $u_i$ and $v_j$, using $u_i + v_j = c_{ij}$

3. Compute the opportunity cost using $\Delta_{ij} = c_{ij} - ( u_i + v_j )$.

4. Check the sign of each opportunity cost.

a) If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimal solution. On the other hand,
b) if one or more unoccupied cell has negative opportunity cost, the given solution is not an optimal solution and further savings in transportation cost are possible.

5. Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in the next solution.

6. Draw a closed path or loop for the unoccupied cell selected in the previous step. Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.

7. Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.

8. Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Now, **add** this quantity to all the cells on the corner points of the closed path marked with plus signs, and **subtract** it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.

9. Repeat the whole procedure until an optimal solution is obtained.

## Degeneracy in Transportation Problem:

If the basic feasible solution of a transportation problem with m origins and n destinations has fewer than (m + n – 1) positive $x_{ij}$ (occupied cells), the problem is said to be a degenerate transportation problem.

Degeneracy can occur at two stages:

1. At the initial solution
2. During the testing of the optimal solution

To resolve degeneracy, we make use of an artificial quantity (*d*). The quantity *d* is assigned to that unoccupied cell, which has the minimum transportation cost.

**Example01:** Find the optimum transportation schedule and minimum total cost of transportation.

| | D1 | D2 | D3 | ai |
|---|---|---|---|---|
| O1 | 10 | 7 | 8 | 40 |
| O2 | 15 | 12 | 9 | 15 |
| O3 | 7 | 8 | 12 | 40 |
| bj | 25 | 55 | 20 | |

**Solution:** Since $\sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j$ = 100. So the given transportation problem is balanced. To find initial basic feasible solution, we apply VAM method. In this method we are to find the difference between two least elements in each row and column.

| | | |
|---|---|---|
| 4<br>10 | 40<br>7 | 5<br>8 |
| 8<br>15 | 4<br>12 | 15<br>9 |
| 25<br>7 | 15<br>8 | 3<br>12 |

We first take the least element cell (3,1) lies on the highest difference column where the demand is 25 units and capacity is 40 units. We choose $x_{31}$=25, the min of these two, to convert into it basic cell, and demand is exhausted, Neglecting that column. We again find the difference and applying the same method to get all initial basic feasible solution. The solutions are $x_{12}=40$, $x_{13}=5$, $x_{23}=15$, $x_{31}=25$, $x_{32}=15$.Here the number of solutions =5= (m+n-1) = (3+3-1). This implies, the solution is non-degenerate soln.

**Optimality test**:

To find the optimal solution we are applying **u-v** method.

| | | | |
|---|---|---|---|
| 4<br>10 | 40<br>7 | 5<br>8 | $U_1=-1$ |
| 8<br>15 | 4<br>12 | 15<br>9 | $U_2=0$ |
| 25<br>7 | 15<br>8 | 3<br>12 | $U_3=0$ |

In this method we first take the basic cell (i,j) where $C_{ij}= u_i+v_j$ We choose u3=0,Then for the cell (3,2) we are applying

$C_{32}=u_3+v_3$

=> $8 = 0 + v_2$

=> $v_2=8$, similarly we get all $u_i$ and $v_j$.

**NET EVALUATION**:-

Again for non-basic cell (r,s) the net evaluation $\Delta_{rs} = C_{rs}-( u_r +v_s)$
For the cell (1,1), we get

$\Delta_{11} = C_{11} - (u_1 +v_1)$
$= 10 - (7 - 1)$
$= 10 - 6$
$= 4$

Similarly we get all $\Delta_{rs}$. Since all $\Delta_{rs} > 0$. Therefore the given T.P has optimum solutions.
Therefore, the min cost of Transportation = 280 + 40 + 135 + 175 + 120 = 750 units

**NOTE:**

i) if all $\Delta_{ij}>0$, then the $sol^n$ is optimal and unique.
ii) If all $\Delta_{ij} >0$ with at least one $\Delta_{ij}=0$, then the $sol^n$ is optimal but not unique.
iii) If at least $\Delta_{ij}<0$, then the solutions not optimal, and we are to seek a new basic feasible solution by loop formation.

**Excises:**

1. Find the optimum Transportation schedule using VAM method and minimum total cost of Transportation.

| Company | Retail | | | Supply |
|---|---|---|---|---|
| | R1 | R2 | R3 | |
| P1 | 10 | 7 | 8 | 45 |
| P2 | 15 | 12 | 9 | 15 |
| P3 | 7 | 8 | 12 | 40 |
| Demand | 25 | 55 | 20 | |

2. Solve the assignment problem:

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | 6 | 5 | 8 | 11 | 16 |
| B | 1 | 13 | 16 | 1 | 10 |
| C | 16 | 11 | 8 | 8 | 8 |
| D | 9 | 14 | 12 | 10 | 16 |
| E | 10 | 13 | 11 | 8 | 16 |

3. Solve the transportation problem using VAM.

| | **D** | **E** | **F** | **G** | **Available** |
|---|---|---|---|---|---|
| **A** | 34 | 13 | 17 | 14 | 250 |
| **B** | 16 | 8 | 14 | 10 | 690 |
| **C** | 21 | 14 | 13 | 4 | 400 |
| Demand | 200 | 225 | 475 | 250 | |

4. Solve the following transportation problem:

| | P | Q | R | S | Availability |
|---|---|---|---|---|---|
| A | 22 | 46 | 16 | 40 | 8 |
| B | 42 | 15 | 50 | 18 | 8 |
| C | 82 | 32 | 48 | 60 | 6 |
| D | 40 | 40 | 36 | 83 | 3 |
| Requirements | 2 | 2 | 5 | 6 | |