Date 13/05/22

Algebraic Extension

If E is an field extension of field F then $a \in E$ is said to be algebraic over the field F if there exist some polynomial $f(x) \in F(x)$ s.t $f(a) = 0$

Example i)

$$\begin{array}{c} \mathbb{C} \\ | \\ \mathbb{R} \end{array} \to \mathbb{R}[x] \to x^2+1$$

$x^2+1=0$
$x = \pm i$

$x = i$

$\mathbb{C} \ni i$ is algebraic over $\mathbb{R}$

$f(x) = x^2+1$

ii) $f(x) = (x+1)(x^2+1)$

Q $1 \in \mathbb{C}$ is algebraic over $\mathbb{R}$?

$f(x) = x-1$

Q

$$\begin{array}{c} \mathbb{R} \ni 1 \\ | \\ \mathbb{Q} \end{array}$$

$\Rightarrow a = 1$ is algebraic over $\mathbb{Q}$? Yes

$f(x) = x-1 \in \mathbb{Q}[x]$ $\quad 1, -1 \in \mathbb{Q}$

$\Rightarrow$ $a = \sqrt{2} \in \mathbb{R}$

$f(x) = x^2-2 \in \mathbb{Q}[x]$ $\quad 1, -2 \in \mathbb{Q}$

$\Rightarrow$ $a = e$

Let $f(x) = x - e$

$(1, -e) \notin \mathbb{Q}$

There does not exist any polynomial $f(x) \in \mathbb{Q}[x]$ s.t $f(e) = 0$, $e \in \mathbb{R} \Rightarrow a = e$ is not algebraic over $\mathbb{Q}$

Date
Page

Q $\begin{array}{c} \mathbb{C} \\ | \\ \mathbb{R} \end{array}$ $\ni e$

$f(x) \; x - e$

$(1, e) \in \mathbb{R}$

So e is algebraic over $\mathbb{R}$

Definition → If E is field extension of F then E is said to be algebraic extension over F if every element in E is algebraic over F.

Ques If $E = \mathbb{R}$, $F = \mathbb{Q}$
Is $\begin{array}{c} \mathbb{R} \\ | \\ \mathbb{Q} \end{array}$ is algebraic extension over $\mathbb{Q}$? No

Logic ⇒ $a = e$ is not algebraic over $\mathbb{Q}$
$a \in \mathbb{R}$
⇒ $\begin{array}{c} \mathbb{R} \\ | \\ \mathbb{Q} \end{array}$ is not algebraic field.

Ques $\begin{array}{c} \mathbb{C} \\ | \\ \mathbb{R} \end{array}$ is algebraic field extension over $\mathbb{R}$? Yes

Logic ⇒ Yes, Every element of $\mathbb{C}$ is an algebraic over $\mathbb{R}$

Result $\Rightarrow$ Every finite field extension is an algebraic extension