APPLICATION OF FOURIER TRANSFORM TO BOUNDARY VALUE PROBLEMS

A bounday value problem is a differential equation together with boundary conditions i.e conditions specified at the extremes of the independent variable in the equation.

A solution of a BVP is solution of the differential equation which also satisfies the boundary conditions.

In one dimensional BVP, the partial diff eq^n. can easily be transformed into O.D.E by applying a suitable transform. The required solution is then obtained by solving this O.D.E. and inverting by means of the inversion formula of the used transform.

Choice of a suitable transform

(i) If in a problem $u(x,t)_{x=0}$ is given, we use infinite sine transform

(ii) If $\left(\frac{\partial u(x,t)}{\partial x}\right)_{x=0}$ is given we use infinite cosine transform

(iii) If $u(0,t)$ and $u(l,t)$ are given we use finite sine transform

(iv) If $\left(\frac{\partial u}{\partial x}\right)_{x=0}$ and $\left(\frac{\partial u}{\partial x}\right)_{x=l}$ are given, then we use finite cosine transform.

Problems. (2)

P.1 Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ if $u(0,t)=0$

$u(x,0) = e^{-x}$ $(x>0)$, $u(x,t)$ is bounded

where $x>0$, $t>0$.

Solution. Given $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $x>0$, $t>0$ — (1)

with boundary conditions (i) $u(0,t)=0$

(ii) $u(x,t)$ is bounded

and initial condition (iii) $u(x,0) = e^{-x}$, $x>0$

since $u(0,t)$ is given. we take fourier sine transform on both sides of (1)

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = 2F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \int_0^\infty \frac{\partial u}{\partial t}\sin px\,dx = 2\left[p\,u(0,t) - p^2\overline{u}_s\right]$$

where $\overline{u}_s = F_s\{u\} = \int_0^\infty u(x,t)\sin px\,dx$

and using $F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = p\,u(0,t) - p^2\overline{u}_s$

$$\Rightarrow \frac{d}{dt}\int_0^\infty u(x,t)\sin px\,dx = -2p^2\overline{u}_s \quad \text{using (i)}$$

$$\Rightarrow \frac{d\overline{u}_s}{dt} = -2p^2\overline{u}_s \quad \text{— (2)}$$

This is O.D.E and its solution is

$$\overline{u}_s(p,t) = \overline{u}_s = c\,e^{-2p^2t} \quad \text{— (3)}$$

Initially at $t=0$ $\overline{u}_s(p,0) = c$

$$\therefore c = \int_0^\infty u(x,0)\sin px\,dx$$

$$= \int_0^\infty e^{-x}\sin px\,dx \quad \text{using (iii)}$$

$$c = \frac{p}{1+p^2} \quad \left(\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right)$$

using it in (3) we have

$$\overline{u}_s = \frac{p}{1+p^2}e^{-2p^2t}$$

Taking inverse Fourier Sine transform ③

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{p}{1+p^2} e^{-2p^2t} \sin px \, dp$$

It gives the required soln. of the given BVP.

P-2

The temperature distribution $u(x,t)$ in a thin, homogeneous, infinite bar can be modelled by the initial BVP

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \; t > 0 \quad \text{— (1)}$$

with (i) $u(x,0) = \begin{cases} 1 & -l < x < l \\ 0 & \text{otherwise} \end{cases}$

(ii) $u(x,t)$ is finite as $x \to \pm\infty$

Solution:- Since domain of bar is $-\infty < x < \infty$, we use the Fourier transform with respect to x.

let $F\{u(x,t)\} = \bar{u}(p,t) = \bar{u}$

Taking Fourier transform on the both sides of ①

$$F\left\{\frac{\partial u}{\partial t}\right\} = F\left\{c^2 \frac{\partial^2 u}{\partial x^2}\right\} = c^2 F\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-ipx} dx = c^2\{-p^2 \bar{u}\}$$

$$\Rightarrow \frac{d}{dt}\int_{-\infty}^{\infty} u(x,t) e^{-ipx} dx = -c^2 p^2 \bar{u}$$

$$\frac{d\bar{u}}{dt} = -c^2 p^2 \bar{u} \quad \text{— (2)} \quad \text{(It is O.D.E)}$$

or $$\frac{d\bar{u}}{\bar{u}} = -c^2 p^2 dt$$

Integrating $\log \bar{u} = -c^2 p^2 t + \log K$

$$\Rightarrow \bar{u} = K e^{-c^2 p^2 t} \quad \text{— (3)}$$

where K is constant

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Initially i.e. at $t=0$

$$\bar{u}(p,0) = K$$

$$\therefore K = F\{u(x,0)\} = \int_{-\infty}^{\infty} u(x,0)\, e^{-ipx}\, dx$$

$$= \int_{-l}^{l} e^{-ipx}\, dx \qquad \text{by (i)}$$

$$K = \frac{e^{ipl} - e^{-ipl}}{ip} = \frac{2\sin pl}{p}$$

from (3) we have

$$\bar{u} = \frac{2\sin pl}{p}\, e^{-c^2p^2t}$$

Taking inverse fourier transform

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin pl}{p}\, e^{-c^2p^2t}\, e^{ipx}\, dp$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin pl}{p}(\cos px + i\sin px)\, e^{-c^2p^2t}\, dp$$

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin pl \cos px}{p}\, e^{-c^2p^2t}\, dp$$

$\left(\because \dfrac{\sin pl \sin px}{p} \text{ is odd } f^n \text{ of } p\right)$

P.3 The steady state temperature distribution $u(x,y)$ in a thin homogeneous semi-infinite plate is governed by the BVP

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0<x<l,\ 0<y<\infty$$

with (i) $u(0,y) = e^{-2y}$, $u(l,y) = 0$, $y>0$

& (ii) $\left(\dfrac{\partial u}{\partial y}\right)_{y=0} = 0$, $0<x<l$.

Find the temperature distribution $u(x,y)$, $0<x<l$, $y>0$.