Equations Reducible to linear form (Bernoulli's equation):-

An equation to the form-

① $\frac{dy}{dx} + Py = Qy^n$ where → $n = 2, 3, 4 ----$
$n \neq 0, 1$

Rule → ① Divide y^n both side

② $y^{-n}\frac{dy}{dx} + Py^{-(n+1)} = Q$ —②

③ Let $t = y^{-n+1}$ [differentiate both side w.r. to x.]

→ $(-n+1)y^{-n+\cancel{1}-\cancel{1}}\frac{dy}{dx} = \frac{dt}{dx}$

→ $y^{-n}\frac{dy}{dx} = \frac{1}{1-x}\frac{dt}{dx}$

Now-② is→

$\frac{1}{1-n}\frac{dt}{dx} + Pt = Q.$

$\frac{dt}{dx} + P(1-n)t = Q(1-n)$ →③ (linear ODE)

Question:- (1) $\frac{dy}{dx} + \frac{y}{x} = y^2 \sin x$ — (1)

Solution:-

dividing the equation by y^2,

$\Rightarrow \frac{1}{y^2} \cdot \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} = \sin x.$

Let →

$\frac{1}{y} = t$

$-\frac{1}{y^2} \times \frac{dy}{dx} = \frac{dt}{dx}.$

$\Rightarrow -\frac{dt}{dx} + \frac{1}{x}(t) = \sin x$

$\Rightarrow \frac{dt}{dx} - \frac{1}{x}(t) = -\sin x$ — (2)

which is ~~a~~ linear ODE

$\frac{dt}{dx} + P' t = Q'$

So, $P' = -\frac{1}{x}$, $Q' = -\sin x$

$\therefore I.F. = e^{\int P' dx} \Rightarrow e^{\int -\frac{1}{x} dx} \Rightarrow e^{-\log x} \Rightarrow e^{\log(1/x)}$

$\left[I.F. = \frac{1}{x} \right]$

Now, solution of (2) is →

$t\left(\frac{1}{x}\right) = \int -\sin x \times \frac{1}{x} dx + C.$

$\left[\therefore t \Rightarrow \frac{1}{y} \right]$

$$\boxed{\frac{1}{xy} = C - \int \frac{\sin x}{x}}$$

Ans

Question (2):- $x\frac{dy}{dx} + y = y^2 \log x$.

Solution:-

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = y^2\left(\frac{\log x}{x}\right)$$

[divide throughout by y^2]

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = \frac{\log x}{x}$$

$$\frac{1}{y} = t$$

$$-\frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow -\frac{dt}{dx} + \frac{1}{x}t = \frac{\log x}{x}$$

$$\frac{dt}{dx} - \frac{1}{x}t = -\frac{\log x}{x} \quad \text{—— (2)}$$

here $\rightarrow P' = -\frac{1}{x}$, $Q' = -\frac{\log x}{x}$

$$\therefore I.f. = e^{\int P'.dx}$$

$$\Rightarrow e^{\int -\frac{1}{x}dx} \Rightarrow e^{-\log x}$$

$[I.f. = 1/x]$

Now $\rightarrow$ $t\left(\frac{1}{x}\right) = \int -\frac{\log x}{x} \times \frac{1}{x}dx + C$

$$\rightarrow t\left(\frac{1}{x}\right) = \int\left(-\frac{1}{x^2}\right)\log x\,dx + C$$

$$\frac{t}{x} = (\log x)\cdot\frac{1}{x} - \int\left(\frac{1}{x}\cdot\frac{1}{x}\right)dx + C$$

$$\frac{t}{x} = \frac{\log x}{x} + \frac{1}{x} + C$$

$$\frac{t}{x} = \frac{\log x}{x} + \frac{1}{x} + C \qquad \left[\because t = \frac{1}{y}\right] \text{Put.}$$

$$\boxed{\frac{1}{xy} = \frac{\log x}{x} + \frac{1}{x} + C}$$

Question-(3) $\Rightarrow 2\frac{dy}{dx} - y\sec x = y^3 \tan x.$

Solution:-

$\rightarrow \frac{2}{y^3}\frac{dy}{dx} - \frac{1}{y^2}\sec x = \tan x.$

$t = -\frac{1}{y^2}$

$\frac{dt}{dx} = \frac{2}{y^3}\frac{dy}{dx}$

$\rightarrow \frac{dt}{dx} + \sec x \cdot t = \tan x$

$P' = \sec x$

$Q' = \tan x$

$\therefore \ I.f = e^{\int P\,dx} \Rightarrow e^{\int \sec x\,dx} \Rightarrow e^{\log(\sec x + \tan x)}$

$[I.f. = \sec x + \tan x]$

Solution is $\rightarrow$

$t(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + c$

$\Rightarrow \int (\sec x \tan x + \sec^2 x - 1)\,dx + C$

$\Rightarrow \sec x + \tan x - x + C.$

$\left[\because t = -\frac{1}{y^2}\right]$

So,

$$\boxed{-\frac{1}{y^2}(\sec x + \tan x) = \sec x + \tan x - x + C}$$

Ans.