Cubic Spline Method → (linear spline).

Let the given data points be,

$(x_i, y_i) \quad i = 0,1,2,---n$ —①

where $a = x_0 < x_1 < x_2 --< x_n = b$

and let $h_i = x_i - x_{i-1}$

$i = 1,2,--n$ —②

further, let $S_i(x)$ be the spline of degree one defined in the interval $[x_{i-1}, x_i]$ obviously $S_i(x)$ represents a straight line joining the points (x_{i-1}, y_{i-1}) and (x_i, y_i)

we can write $S_i(x) = y_{i-1} + m_i(x - x_{i-1})$ —③

where $m_i = \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$ —④

Setting $i = 1,2,--n$ successively in eq. ③ we obtain different splines of degree one valid in the subintervals 1 to n.

Cubic Spline - let $S_i(x)$ be the cubic spline defined in the interval $[x_{i-1}, x_i]$. the conditions for the natural cubic splines are,

① $S_i(x)$ is almost a cubic in each sub-interval $[x_{i-1}, x_i]$, $i = 1,2,--n$

② $S_i(x_i) = y_i$, $i = 0,1,2,--n$

③ $S_i(x)$, $S_i'(x)$ and $S_i''(x)$ are continuous in $[x_0, x_n]$ and

④ $S_i''(x_0) = S_i''(x_n) = 0$

the recurrence relation for this method

$$\frac{h_i}{6} M_{i-1} + \frac{1}{3}(h_i + h_{i+1}) M_i + \frac{h_{i+1}}{6} M_{i+1}$$

$$= \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \quad (i = 1, 2, --n-1) \quad —①$$

For equal intervals, we have $h_i = h_{i+1} = h$

and eq. ① simplifies to.

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

$$(i = 1, 2, ---n-1) \quad —②$$

The equation of the cubic spline in the subinterval (x_i, x_{i+1}).

$$S_{i+1}(x) = \frac{1}{h_{i+1}}\left[\frac{(x_{i+1}-x)^3}{6} M_i + \frac{(x-x_i)^3}{6} M_{i+1} + \left(y_i - \frac{h_{i+1}^2}{6} M_i\right)(x_{i+1}-x) + \left(y_{i+1} - \frac{h_{i+1}^2}{6} M_{i+1}\right)(x - x_i)\right]$$

—③

Example ① We consider the function

$y(x) = \sin x$ in $[0, \pi]$

Here $M_0 = M_N = 0$

$N = 2$, $h = \frac{\pi}{2}$

then $y_0 = y_2 = 0$ $y_1 = 1$

and $M_0 = M_2 = 0$

Now by eq. (2), we obtain-

$$M_0 + 4M_1 + M_2 = \frac{6}{h^2}(y_0 - 2y_1 + y_2)$$

or $M_1 = -\frac{12}{\pi^2}$

Now by eq. (3) gives the spline in each interval, thus, in $0 \le x \le \pi/2$ we find.

$$S(x) = \frac{2}{\pi}\left(\frac{-2x^3}{\pi^2} + \frac{3x}{2}\right)$$

which gives $S'(x) = \frac{2}{\pi}\left[\frac{-2 \times 3x^2}{\pi^2} + \frac{3}{2}\right]$ —①

$$S'\left(\frac{\pi}{4}\right) = \frac{2}{\pi}\left[\frac{-6}{\pi^2}\left(\frac{\pi^2}{16}\right) + \frac{3}{2}\right] = \frac{9}{4\pi} = 0.71619$$

exact value of $S'\left(\frac{\pi}{4}\right) = 0.70710$.

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

the % error in the computed value of $S'\left(\frac{\pi}{4}\right)$ is 1.28 %.

from ① $S''(x) = \frac{-24}{\pi^3}x$.

$$S''\left(\frac{\pi}{4}\right) = -0.60792$$

% error = 14.03 % in this result.

Note → As we have to calculate the derivative of sin x in the interval 0 – 90° we can break this interval with 10° (step size).