Q- Use dynamic programming to find the value of

Max $z = y_1 y_2 y_3$

s.t.

$$y_1 + y_2 + y_3 = 5$$

$$y_1, y_2, y_3 \geq 0$$

Solution: This is three stage problem, because number of variables are three.

Let S_1, S_2 and S_3 are state variables.

$$S_3 = y_1 + y_2 + y_3 \quad \text{———} \quad (1)$$

$$S_2 = S_3 - y_3 = y_1 + y_2 \quad \text{———} \quad (2)$$

$$S_1 = S_2 - y_2 = y_1 \quad \text{———} \quad (3)$$

The General functional eqn

$$F_j(S_j) = \max_{y_j} \left[f_j(y_j)\, F_{j-1}(S_{j-1}) \right], \quad j = n, n-1, 2$$

$$F_1(S_1) = f_1(y_1)$$

Now,

$$F_3(S_3) = \max_{y_3} \left[y_3 \cdot F_2(S_2) \right]$$

$$F_2(S_2) = \max_{y_2} \left[y_2 \cdot F_1(S_1) \right] = \max_{y_2} \left[y_2 \cdot (S_2 - y_2) \right]$$

$$F_1(S_1) = y_1 = S_2 - y_2 \quad \text{from (3)}$$

Find partial derivative of $F_2(S_2)$ w.r.t. y_2

$$\frac{\partial F_2(S_2)}{\partial y_2} = 0$$

$$y_2(-1) + (S_2 - y_2) = 0 \Rightarrow 2y_2 = S_2$$

$$\Rightarrow y_2 = \frac{S_2}{2} \quad \text{———} \quad (4)$$

$$F_2(s_2) = \max_{y_2} \left[\frac{s_2}{2} \left(s_2 - \frac{s_2}{2} \right) \right]$$

$$F_2(s_2) = \frac{s_2^2}{4} \qquad \because \; y_2 = \frac{s_2}{2}$$

$$F_3(s_3) = \max_{y_3} \left[y_3 \cdot F_2(s_2) \right]$$ Bellman's principle of optimality

$$= \max_{y_3} \left[y_3 \cdot \frac{s_2^2}{4} \right]$$

$$= \max_{y_3} \left[y_3 \cdot \frac{(s_3 - y_3)^2}{4} \right]$$ from (2)

Find partial derivative of $F_3(s_3)$ w.r.t. y_3,

$$\frac{\partial F_3(s_3)}{\partial y_3} = 0$$

$$y_3 \cdot \frac{2(s_3 - y_3)(-1)}{4} + \frac{(s_3 - y_3)^2}{4} = 0$$

$$\Rightarrow \frac{(s_3 - y_3)^{\cancel{2}}}{\cancel{4}} = \frac{2y_3 \cancel{(s_3 - y_3)}}{\cancel{4}}$$

$$s_3 - y_3 = 2y_3$$

$$s_3 = 3y_3$$

$$y_3 = \frac{s_3}{3} = \frac{5}{3}$$

$$y_2 = \frac{s_2}{2} \Rightarrow y_2 = \frac{1}{2} \times \frac{10}{3} = 5/3$$

$$s_2 = s_3 - y_3$$

$$s_2 = 5 - \frac{5}{3} = \frac{10}{3}$$

$$s_1 = y_1 = s_2 - y_2 = \frac{10}{3} - \frac{5}{3} = \frac{5}{3}$$

Hence $y_1 = y_2 = y_3 = 5/3$

and $\max y_1 y_2 y_3 = \frac{5}{3} \times \frac{5}{3} \times \frac{5}{3} = \frac{125}{27}$