Finite Difference →

Interpolation → $y = f(x)$ $x_0 \le x \le x_n$

Assuming $f(x)$ is single valued and continuous and it is known explicitly, then the values of $f(x)$ corresponding to certain given values of x, say $x_0, x_1, - x_n$ can easily be computed and tabulated. Given the set of tabular values (x_0, y_0). (x_1, y_1) $(x_2, y_2$ and (x_n, y_n) satisfying the relation $y = f(x)$ where the explicit nature of $f(x)$ is not known, it is required ~~to~~ find a simpler function, say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. Such a process is called interpolation. If $\phi(x)$ is a polynomial, then the process is called polynomial interpolation and $\phi(x)$ is called the interpolating polynomial. Similarly different types of interpolation arise depending on whether $\phi(x)$ is a finite trigonometric series.

If $f(x)$ is continuous in $x_0 \le x \le x_n$, ~~then~~ given $\varepsilon > 0$, there exists a polynomial $P(x)$ such that,

$$|f(x) - P(x)| < \varepsilon$$

for all x in (x_0, x_n).

this means that it is possible to find a polynomial $P(x)$ whose graph remains within the region bounded by $y = f(x) - \varepsilon$ and $y = f(x) + \varepsilon$ for all x b/w x_0 and x_n, however small ε may be.

Finite Differences → Assume that we have a table of values (x_i, y_i) $i = 0, 1, 2, \dots n$ of any function $y = f(x)$, the values of x being equally spaced i.e. $x_i = x_0 + ih$
$i = 0, 1, 2 \dots n$

Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x, or to obtain the derivative of $f(x)$ for some x in the range $x_0 \le x \le x_n$.

Forward Differences → If $y_0, y_1, y_2 \dots y_n$ denote a set of values of y then $y_1 - y_0$, $y_2 - y_1$ … $y_n - y_{n-1}$ are called the differences of y.

Denoting these differences by $\Delta y_0, \Delta y_1, \dots \Delta y_{n-1}$ we have,

$\Delta y_0 = y_1 - y_0$, $\Delta y = y_2 - y_1$, $\Delta y_{n-1} = y_n - y_{n-1}$,

where Δ is called the forward difference operator and $\Delta y_0, \Delta y_1$ are called first forward differences. the differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1 \dots$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$= y_2 - y_1 - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$= y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Forward Difference Table

x	y_0	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Backward Differences $\rightarrow$ the differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called first backward differences if they are denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively, so that -

$\nabla y_1 = y_1 - y_0 \quad \nabla y_2 = y_2 - y_1$

$\nabla y_n = y_n - y_{n-1}$

where ∇ is called the backward difference operator. In the similar manner we can define backward differences of higher orders.

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$
$$= y_2 - y_1 - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0$$
$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
		∇y_1					
x_1	y_1		$\nabla^2 y_2$				
		∇y_2		$\nabla^3 y_3$			
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
		∇y_3		$\nabla^3 y_4$			
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	$\nabla^5 y_5$	
		∇y_4		$\nabla^3 y_5$			
x_4	y_4		$\nabla^2 y_5$		$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$
		∇y_5		$\nabla^3 y_6$			
x_5	y_5		$\nabla^2 y_6$				
		∇y_6					
x_6	y_6						

Central Differences -

The central difference operator δ is defined by the relations.

$$y_1 - y_0 = \delta y_{1/2}$$

$$y_2 - y_1 = \delta y_{3/2}$$

$$y_n - y_{n-1} = \delta y_{n-1/2}$$

Similarly, we can define higher-order central differences.

Central Difference Table -

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^3 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^3 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^3 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						