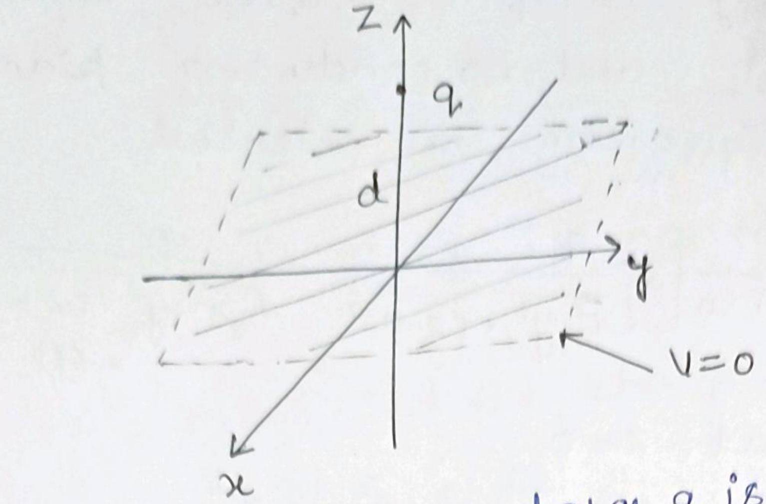
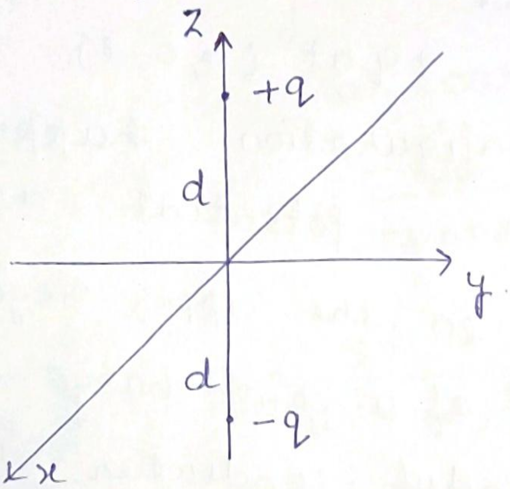
the method of images

the classic image problem:-



Suppose a point charge q is held at a distance d above an infinite grounded conducting plane.



for this we solve Poisson's equation in the region $z>0$, with a single point charge q at (0,0,d) ∴Here are the boundary conditions.

① $V=0$ when $z=0$ (Conducting plane is grounded)

② $V\to 0$ far from the charge ($x^2+y^2+z^2 \gg d^2$)

By uniqueness theorem it is guaranteed that there is only one function that meets these requirements.

Now we try a new problem consists of two point charges $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$ and no conducting plane.

For this configuration the potential:

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] \quad -(1)$$

$V = 0$ at $z = 0$

$V \to 0$ for $x^2+y^2+z^2 \gg d^2$.

the only charge in the region $z>0$ is the point charge $+q$ at $(0,0,d)$.

the second configuration happens to produce exactly the same potential as the first configuration in the upper region $z \geq 0$.

the potential of a point charge above an infinite grounded conductor is given by eq. (1).

Induced Surface Charge →

the surface charge induced on the conductor.

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \quad -(2)$$

$\frac{\partial V}{\partial n}$ → normal derivative of V at the surface.

Normal direction is z so eq.(2) becomes

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} \quad -(3)$$

Now By equation (1)

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{\{x^2+y^2+(z-d)^2\}^{3/2}} + \frac{q(z+d)}{\{x^2+y^2+(z+d)^2\}^{3/2}}\right\}$$

$$\sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}} \quad -(4)$$

induced charge is negative type and greatest at $x=y=0$.

the total induced charge,

$$Q = \int \sigma\, da$$

$da = dx\,dy$, (r, ϕ) $r^2 = (x^2+y^2)$

$da = r\,dr\,d\phi$

in polar coordinates.

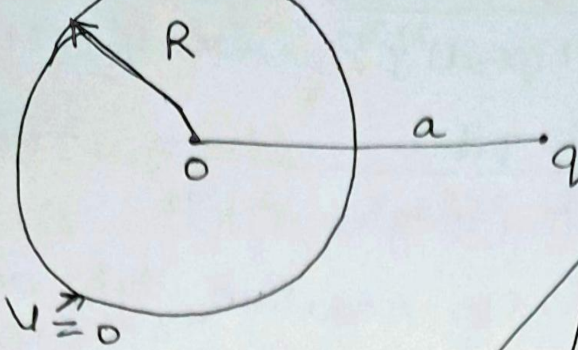
then $\sigma(R) = \dfrac{-qd}{2\pi(r^2+d^2)^{3/2}}$

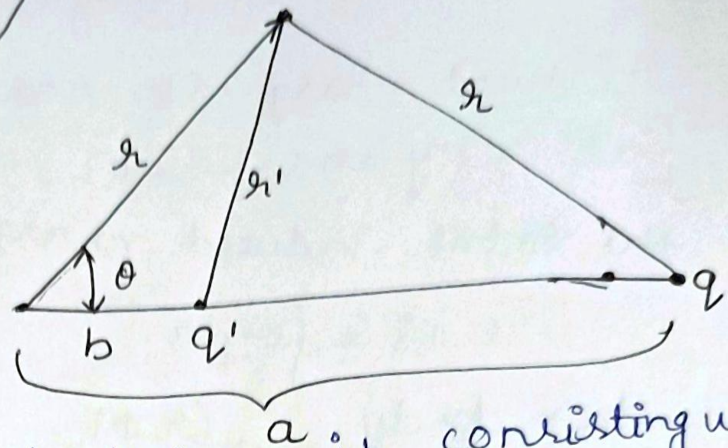
$$Q = \int_0^{2\pi}\int_0^{\pi} \frac{-qd}{2\pi(r^2+d^2)^{3/2}}\, r\,dr\,d\phi$$

$$Q = \frac{qd}{(r^2+d^2)^{1/2}}\Bigg|_0^{\infty} = -q \quad -(5)$$

the total charge induced on the plane is $-q$.

Q. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere.





<u>Solution</u> → A point charge is consisting with another point charge $q' = -\frac{R}{a} q$ —(1)

placed at a distance $b = \frac{R^2}{a}$ —(2)

the potential of the new configuration is,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right) \quad \text{—(3)}$$

<u>Force and Energy</u> → the charge q is attracted toward the plane, because the negative induced charge.

Force $f = -\frac{1}{4\pi\epsilon_0} \frac{(q)^2}{(2d)^2} \hat{z}$ —(4)

$W = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{2d}$ (with the two point charges and no conductors)

for a single charge and conducting plane the energy is half.

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d} \quad -(5)$$

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau$$

in first case $\begin{cases} z>0 & \text{for the upper region.} \\ z<0 & \text{for the lower region.} \end{cases}$

in second case only upper region

we calculate contains a nonzero field.

energy

$$W = \int_{\infty}^{d} F \cdot dl = \frac{1}{4\pi\epsilon_0}\int_{\infty}^{d} \frac{q^2}{4z^2}\, dz$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{-q^2}{4z}\right)\Big|_{\infty}^{d}$$

$$\boxed{W = \frac{-1}{4\pi\epsilon_0}\frac{q^2}{4d}}$$

Ans.