Newton's Formula for interpolation →

Given the set of $(n+1)$ values viz. (x_0, y_0) (x_1, y_1) (x_2, y_2) --- (x_n, y_n) of x and y. it is required to find $y_n(x)$ a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistant

let $x_i = x_0 + ih \qquad i = 0, 1, -- n$

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + -- + a_n(x - x_0)(x - x_1) --- (x_n - x_{n-1}) \quad -①$$

y and $y(x)$ should agree at the set of tabulated points

$y_1 - y_0 = \Delta y_0$

$y_1 - y_0 = \nabla y_1$

$a_0 = y_0$

$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h},$$

$$a_2 = \frac{\Delta^2 y_0}{h^2 2!}, \quad a_3 = \frac{\Delta^3 y_0}{h^3 3!}, \quad a_n = \frac{\Delta^n y_0}{h^n n!}$$

Setting $x = x_0 + ph$ and substituting for a_0, a_1 -- a_n, eq. ① gives.

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2) -- (p-n+1)}{n!}\Delta^n y_0 \quad -②$$

which is Newton's forward difference interpolation formula and is useful for interpolation near the beginning of a set of tabular values.

Expanding $y(x+h)$ by Taylor's series.

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \cdots$$

Neglecting the terms containing h^2 and higher powers of h this gives.

$$y'(x) \simeq \frac{1}{h}[y(x+h) - y(x)]$$

$$= \frac{1}{h}\Delta y(x).$$

$$Dy(x) = y'(x)$$

$$D \equiv \frac{1}{h}\Delta, \quad D^{n+1} = \frac{1}{h^{n+1}}\Delta^{n+1}$$

we thus obtain,

$$y^{(n+1)}(x) \simeq \frac{1}{h^{n+1}}\Delta^{n+1}y(x) \quad (3)$$

the error commited in replacing the function $y(x)$ by means of the polynomial $y_n(x)$,

$$y(x) - y_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!}y^{n+1}(\xi)$$

$$x_0 < \xi < x_n \quad (4)$$

So eq. (4) changes by eq. (3) as.

$$y(x) - y_n(x) = \frac{p(p-1)(p-2)\cdots(p-n)}{(n+1)!}\Delta^{n+1}y(\xi) \quad (5)$$

In eq. (1) we choose another form,

$$y_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1})$$
$$+ a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \cdots$$
$$+ a_n(x-x_n)(x-x_{n-1})\cdots(x-x_1).$$

and then impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, \ldots x_2, x_1, x_0$ we obtain;

$$\boxed{y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\cdots(p+n-1)}{n!}\nabla^n y_n \quad \text{where } p = \left(\frac{x-x_n}{h}\right)} \quad -(6)$$

this is Newton's backward difference interpolation formula and it uses tabular values to the left of y_n. this formula is useful for interpolation near the end of the tabular values.

the error in this formula is written as,

$$\boxed{y(x) - y_n(x) = \frac{p(p+1)(p+2)\cdots(p+n)}{(n+1)!}\nabla^{n+1} y(\xi)} \quad -(7)$$

$$\left.\begin{array}{l} x_0 < x < x_n \\ x = x_n + ph \end{array}\right\}$$

Gauss's Central Difference Formulae →

Gauss's forward formula.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

the differences used in this formula lie on the line shown in table which has a form.

$$y_p = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} + \cdots \quad (8)$$

$G_1, G_2 \rightarrow$ have to be determined

the y_p on the left side can be expressed in terms of $y_0, \Delta y_0 \cdots$

$$y_p = E^p y_0 = (1+\Delta)^p y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \quad (9)$$

similarly for right side of eq. (8).

$$\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0$$

$$= \Delta^2 (1+\Delta)^{-1} y_0$$

$$= \Delta^2 (1 - \Delta + \Delta^2 - \Delta^3 + \cdots) y_0$$

$$= \Delta^2 (y_0 - \Delta y_0 + \Delta^2 y_0 - \Delta^3 y_0 + \cdots)$$

$$= \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots$$

$$\Delta^3 y_{-1} = \Delta^3 E^{-1} y_0$$

$$= \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0$$

$$\Delta^2 y_{-2} = \Delta^4 E^{-2} y_0$$

$$= \Delta^4 (1+\Delta)^{-2} y_0$$

$$= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0$$

Hence eq. (8)

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$= y_0 + G_1 \Delta y_0 + G_2 (\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots)$
$+ G_3 (\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0)$
$+ G_4 (\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \cdots)$ — (10)

Equating the coefficients of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ on both sides of eq. (10) we find that.

$$\left.\begin{aligned} G_1 &= p \\ G_2 &= \frac{p(p-1)}{2!} \\ G_3 &= \frac{(p+1)p(p-1)}{3!} \\ G_4 &= \frac{(p+1)p(p-1)(p-2)}{4!} \end{aligned}\right\} \text{— (11)}$$

Gauss's backward formula →

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
$\vdots$							
x_{-1}	y_{-1}						
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1						
$\vdots$	$\vdots$						

Gauss's backward formula can be assumed tobe of the form.

$$y_b = y_0 + G_1' \Delta y_{-1} + G_2' \Delta^2 y_{-1} + G_3' \Delta^3 y_{-2} + G_4' \Delta^4 y_{-2} \quad \text{— (12)}$$

Same procedure as in Gauss's forward formula, we find that,

$$\left.\begin{aligned} G_1' &= p, & G_3' &= \frac{(p+1)p(p-1)}{3!} \\ G_2' &= \frac{p(p+1)}{2!}, & G_4' &= \frac{(p+2)(p+1)p(p-1)}{4!} \end{aligned}\right\} \text{(13)}$$