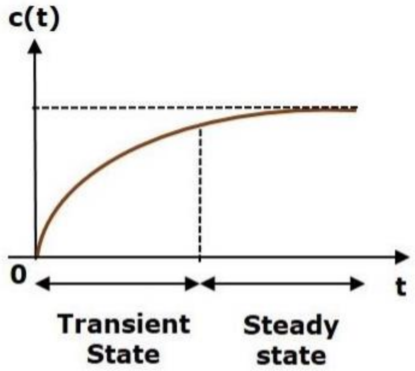
TIME RESPONSE ANALYSIS

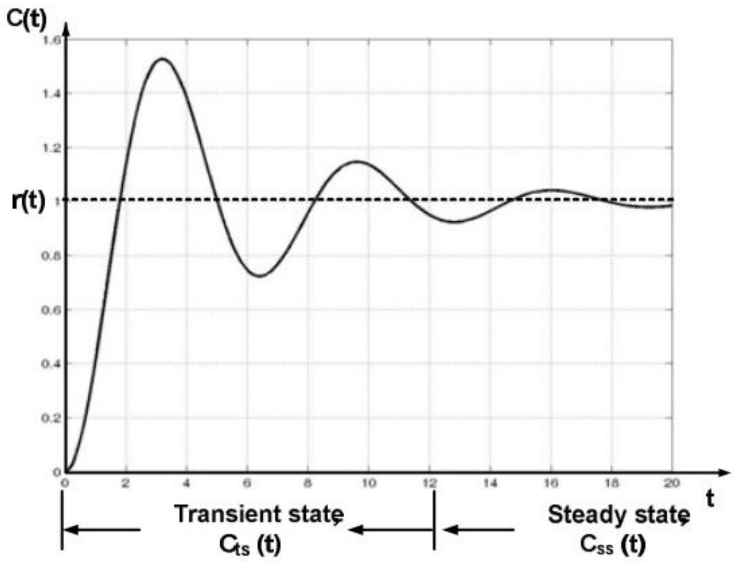
We can analyse the response of the control systems in both the time domain and the frequency domain. Let us now discuss about the time response analysis of control systems.

What is Time Response?

If the output of control system for an input varies with respect to time, then it is called the time response of the control system. The time response consists of two parts.

- Transient response
- Steady state response





The response of control system in time domain is shown in the above figure.

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response.**

The transient response will be zero for large values of 't'. Ideally, this value of 't' is infinity and practically, it is five times constant.

Mathematically, we can write it as-

$$\lim_{t\to\infty} c_{tr}(t) = 0$$

The part of the time response that remains even after the transient response has zero value for large values of 't' is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example - Let us find the transient and steady state terms of the time response of the control system

$$c(t) = 10 + 5e^{-t}$$

Here, the second term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

Standard Test Signals

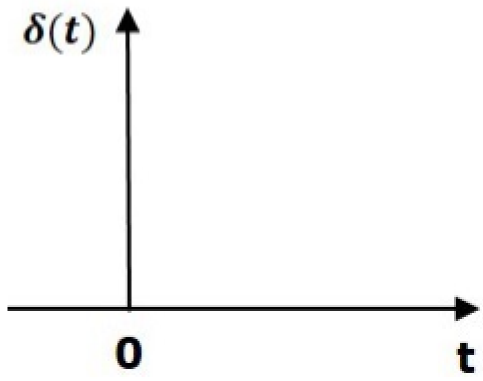
The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

Unit Impulse Signal

A unit impulse signal, δ(t) is defined as-

$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\text{and } \int_{0^-}^{0^+} \delta(t)dt = 1$$



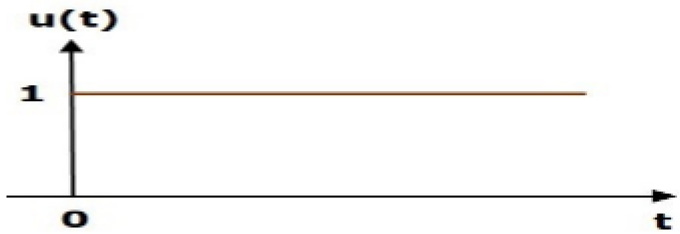
The above figure shows the unit impulse signal

So, the unit impulse signal exists only at ‘t’ is equal to zero. The area of this signal under small interval of time around ‘t’ is equal to zero is one. The value of unit impulse signal is zero for all other values of ‘t’.

Unit Step Signal

A unit step signal, u(t) is defined as-

$$u(t) = 1; t \geq 0$$
$$= 0; t < 0$$



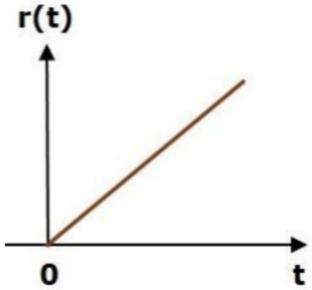
The above figure shows unit step signal.

So, the unit step signal exists for all positive values of‘t’ including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of‘t’.

Unit Ramp Signal

A unit ramp signal, r (t) is defined as-

$$r(t) = t; t \geq 0$$
$$= 0; t < 0$$



the unit ramp signal

We can write unit ramp signal, $r(t)$ in terms of unit step signal, $u(t)$ as

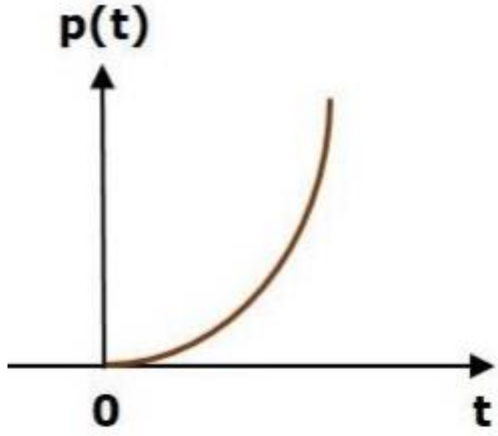
$$r(t) = tu(t)$$

So, the unit ramp signal exists for all positive values of‘t’ including zero. And its value increases linearly with respect to‘t’ during this interval. The value of unit ramp signal is zero for all negative values of‘t’.

Unit Parabolic Signal

A unit parabolic signal, p(t) is defined as,-

$$p(t) = \frac{t^2}{2}; t \geq 0$$
$$= 0; t < 0$$



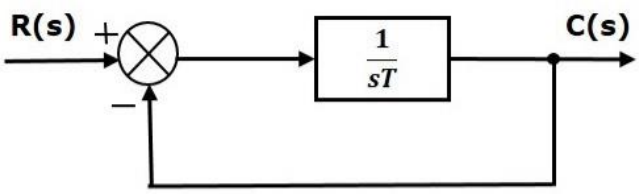
We can write unit parabolic signal, $p(t)$ in terms of the unit step signal, $u(t)$ as,

$$p(t) = \frac{t^2}{2} u(t)$$

So, the unit parabolic signal exists for all the positive values of‘t’ including zero. And its value increases non-linearly with respect to‘t’ during this interval. The value of the unit parabolic signal is zero for all the negative values of‘t’.

Time response of the first order system-

let us discuss the time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1/sT is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1 + \frac{1}{sT}} = \frac{1}{sT + 1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system.**

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT + 1}\right) R(s)$$

Where,

- C(s) is the Laplace transform of the output signal c(t),
- R(s) is the Laplace transform of the input signal r(t),
- T is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal $r(t)$.
- Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$
- Substitute $R(s)$ value in the above equation.
- Do partial fractions of $C(s)$ if required.
- Apply inverse Laplace transform to $C(s)$.

Impulse Response of First Order System

Consider the unit impulse signal as an input to the first order system.

So, r(t)=δ(t)

Apply Laplace transform on both the sides.

R(s) =1

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT + 1}\right)(1) = \frac{1}{sT + 1}$$

Rearrange the above equation in one of the standard forms of Laplace transforms.

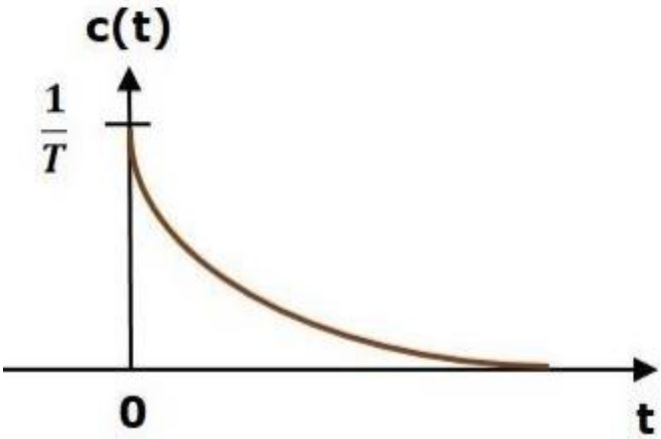
$$C(s) = \frac{1}{T\left(s + \frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s + \frac{1}{T}}\right)$$

Applying Inverse Laplace Transform on both the sides,

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

The unit impulse response is shown in the following figure.



The unit impulse response, c(t) is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

Step Response of First Order System

Consider the unit step signal as an input to first order system.

So, r(t)=u(t)

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\left(sT+1\right)}$$

Do the partial fraction of C(s),

$$C(s) = \frac{1}{s\left(sT+1\right)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s\left(sT+1\right)} = \frac{A\left(sT+1\right)+Bs}{s\left(sT+1\right)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1=A(sT+1)+Bs$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the s terms on both the sides.

$$0=T+B$$

$$\Rightarrow B=-T$$

Substitute, A = 1 and B = −T in partial fraction expansion of C(s)

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The unit step response, c(t) has both the transient and the steady state terms.

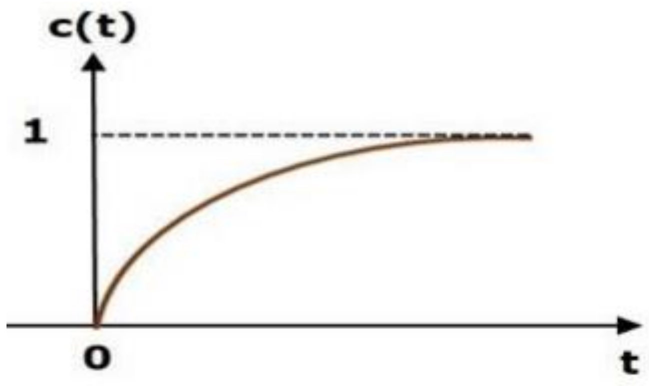
The transient term in the unit step response is -

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)}u(t)$$

The steady state term in the unit step response is –

$$c_{ss}(t) = u(t)$$

The following figure shows the unit step response



The value of the unit step response, c(t) is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Ramp Response of First Order System

Consider the unit ramp signal as an input to the first order system.

So, r(t)=t u(t)

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^2}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s^2}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1}$$

$$\Rightarrow \frac{1}{s^2(sT+1)} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs(sT+1) + Cs^2$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the s terms on both the sides.

$$0=T+B \Rightarrow B=-T$$

Similarly, substitute B = −T and equate the coefficient of s^2 terms on both the sides. You will get $C=T^2$

Substitute A = 1, B = −T and $C=T^2$ in the partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The unit ramp response, c(t) has both the transient and the steady state terms.

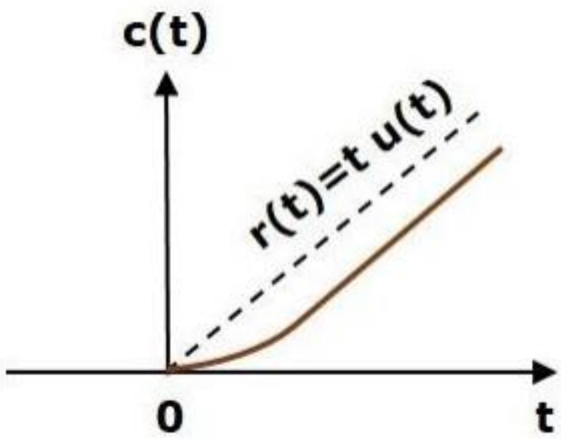
The transient term in the unit ramp response is

$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)}u(t)$$

The steady state term in the unit ramp response is –

$$c_{ss}(t) = (t-T)u(t)$$

The figure below is the unit ramp response:



The unit ramp response, c(t) follows the unit ramp input signal for all positive values of t. But, there is a deviation of T units from the input signal.

Parabolic Response of First Order System

Consider the unit parabolic signal as an input to the first order system.

$$\text{So, } r(t) = \frac{t^2}{2}u(t)$$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute $R(s) = \frac{1}{s^3}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$

Do the partial fraction of C(s)

$$C(s) = \frac{1}{s^3(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT+1}$$

After simplifying, you will get the values of A, B, C, and D as 1, -T, T^2 and $-T^3$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{sT+1} \Rightarrow C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The unit parabolic response, c(t) has both the transient and the steady state terms.

The transient term in the unit parabolic response is-

$$C_{tr}(t) = -T^2 e^{-\left(\frac{t}{T}\right)} u(t)$$

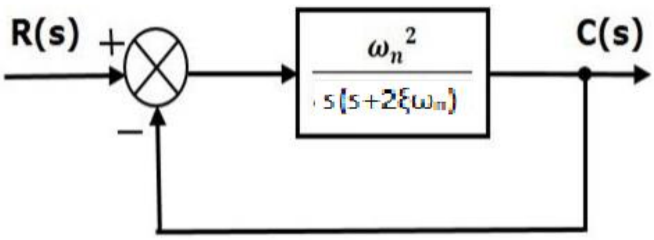
The steady state term in the unit parabolic response is-

$$C_{ss}(t) = \left(\frac{t^2}{2} - Tt + T^2\right) u(t)$$

From these responses, we can conclude that the first order control systems are **not stable with the ramp and parabolic inputs** because these responses go on increasing even at infinite amount of time. The first order control systems are **stable with impulse and step inputs** because these responses have bounded output. But, the **impulse response doesn't have steady state term**. So, **the step signal is widely used in the time domain for analyzing** the control systems from their responses.

Time response of second order system-

Consider the following block diagram of closed loop control system. Here, an open loop transfer function, $\omega_n^2 / s(s+2\xi\omega_n)$ is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute the G(s) value in the above equation-

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where ω_n, the **undamped natural frequency**; and **ξ**, **the damping ratio** of the system.

The damping ratio **ξ** is the **ratio of the actual damping to the critical damping**.

The dynamic behavior of the second-order system can then be described in terms of two parameters **ξ** and **ω_n**.

• **If 0< ξ <1**, the closed-loop **poles are complex conjugates** and **lie in the left-half s plane**.

The system is then called **underdamped**.

• The transient response is **oscillatory**, **If ξ =0**, the transient response does not die out.

• **If ξ =1**, the system is called **critically damped**.

• **Overdamped** systems correspond to **ξ >1**.

The power of ‘s’ is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be **the second order system**.

The characteristic equation is –

$$S^2+2\xi w_n S+w_n^2=0$$

$$\Rightarrow s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

- The two roots are imaginary when ξ = 0.
- The two roots are real and equal when ξ = 1.
- The two roots are real but not equal when ξ > 1.
- The two roots are complex conjugate when 0 < ξ < 1.

Case:-1-Underdamped case (0< ξ <1):

C(s)/R(s) can be written as-

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

The roots for a standard second order system can be calculated as **$S1, S2$ = − $\xi\omega_n \pm j\omega_d$**

Where **. Tł** $= \omega_n\sqrt{1-\xi^2}$ The frequency ω_n is called the damped natural frequency.

We can write C(s) equation as,

$$C(s) = \left(\frac{w_n^2}{s^2+2\xi w_n s+w_n^2}\right)R(s)$$

Where,

- C(s) is the Laplace transform of the output signal, c(t)
- R(s) is the Laplace transform of the input signal, r(t)
- $\boldsymbol{\omega_n}$ is the natural frequency
- $\boldsymbol{\xi}$ is the damping ratio.

Follow these steps to get the response (output) of the second order system in the time domain.

Take Laplace transform of the input signal, r(t).

Consider the equation, $C(s) = \left(\frac{w_n^2}{s^2+2\xi w_n s+w_n^2}\right)R(s)$

Substitute R(s) value in the above equation.

Do partial fractions of C(s) if required.

Apply inverse Laplace transform to C(s).

Step Response of Second Order System

Consider the unit step signal as an input to the second order system. Laplace transform of the unit step signal is,

R(s) = 1/s

We know the transfer function of the second order closed loop control system is,

$$C(s) = \left(\frac{w_n^2}{s^2+2\xi w_n s+w_n^2}\right)R(s)$$

Case 1: $\xi = 0$

Substitute, $\xi = 0$ in the transfer function,

$$C(s) = \left(\frac{w_n^2}{s^2+w_n^2}\right)R(s)$$

Substitute, R(s) = 1/s in the above equation,

$$C(s) = \left(\frac{w_n^2}{s^2+w_n^2}\right)\left(\frac{1}{s}\right)$$

Apply inverse Laplace transform on both the sides.

$$C(t) = (1-\cos(w_n t))u(t)$$

So, the unit step response of the second order system when **ξ=0,** will be a continuous time signal with constant amplitude and frequency.

Case 2: $\xi = 1$

Substitute, $\xi = 1$ in the transfer function,

$$C(s) = \left(\frac{w_n^2}{s^2+2w_n s+w_n^2}\right)R(s)$$

$$C(s) = \left(\frac{w_n^2}{(s+w_n)^2}\right)R(s)$$

Substitute, R(s) = 1/s in the above equation,

$$C(s) = \left(\frac{w_n^2}{(s+w_n)^2}\right)\left(\frac{1}{s}\right)$$

Do partial fractions of C(s)

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, we will get the values of A, B and C as 1, -1 and $-w_n$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: 0 < $\xi < 1$

We can modify the denominator term of the transfer function as follows-

$S^2+2\xi w_n s+w_n^2 = \{ S^2+2\xi w_n s+w_n^2 + (\xi w_n)^2 - (\xi w_n)^2 \}$

$= (s+\xi w_n)^2 + w_n^2(1-\xi^2)$

The transfer function becomes,

$$C(s) = \left(\frac{w_n^2}{(s+\xi wn)^2 + wn^2(1-\xi^2)}\right)R(s)$$

Substitute, R(s) = 1/s in the above equation.

$$C(s) = \left(\frac{w_n^2}{(s+\xi wn)^2 + wn^2(1-\xi^2)}\right)\left(\frac{1}{s}\right)$$

Do the partial fractions of C(s)

$$C(s) = \left(\frac{w_n^2}{(s+\xi wn)^2 + wn^2(1-\xi^2)}\right)\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{Bs+C}{(s+\xi wn)^2 + wn^2(1-\xi^2)}$$

After simplifying, you will get the values of A, B and C as 1, -1 and $-2\xi wn$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s+2\xi wn}{(s+\xi wn)^2 + wn^2(1-\xi^2)}$$

$$= \frac{1}{s} - \frac{s+\xi wn}{(s+\xi wn)^2 + (1-\xi^2)wn^2} - \frac{\xi wn}{(s+\xi wn)^2 + wn^2(1-\xi^2)}$$

$$= \frac{1}{s} - \frac{s+\xi wn}{(s+\xi wn)^2 + (1-\xi^2)wn^2} - \left(\frac{wn.(1-\xi^2)^{1/2}}{(s+\xi wn)^2 + (1-\xi^2)wn^2}\right)\left(\frac{\xi}{(1-\xi^2)^{1/2}}\right)$$

Substitute, $\omega_d = \omega_n\sqrt{1-\xi^2}$. In the above equation.

$$C(s) = \frac{1}{s} - \frac{s+\xi wn}{(s+\xi wn)^2 + wd^2} - \left(\frac{wd}{(s+\xi wn)^2 + wd^2}\right)\left(\frac{\xi}{(1-\xi^2)^{1/2}}\right)$$

Apply inverse Laplace transform on both the side.

$$C(t) = (1 - e^{-\xi wnt}\cos(w_d t) - \frac{\xi}{(1-\xi^2)^{1/2}}(e)^{-\xi wnt}\sin(wdt))u(t)$$

$$C(t) = \{1 - \frac{(e)^{-\xi wnt}}{(1-\xi^2)^{1/2}}((1-\xi^2)^{1/2}\cos(w_d t) + \xi\sin(wdt))\}u(t)$$

If $(1-\xi^2)^{1/2} = \sin(\varnothing)$, then '$\xi$' will be $\cos(\varnothing)$. Substitute these values in the above equation.

$$C(t) = \{1 - \frac{(e)^{-\xi wnt}}{(1-\xi^2)^{1/2}}(\sin(\varnothing)\cos(w_d t) + \cos(\varnothing)\sin(wdt))\}u(t)$$

$$C(t) = \{1 - \frac{(e)^{-\xi wnt}}{(1-\xi^2)^{1/2}}(\sin(\varnothing + w_d t)\}u(t)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'ξ' lies between zero and one.

The error signal for this system is the difference between the input and output and is –

e(t) = r(t) – c(t)

$$= e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right), \quad \text{for } t \geq 0$$

This error signal exhibits a damped sinusoidal oscillation. At steady state, or at t=∞, no error exists between the input and output.

If the damping ratio ξ is equal to zero, the response becomes undamped and oscillations continue indefinitely. The response c(t) for the zero damping case may be obtained by substituting ξ =0

$$c(t) = 1 - \cos\omega_n t, \qquad \text{for } t \geq 0$$

Case 4: $\xi > 1$

We can modify the denominator term of the transfer function as follows-

$S^2+2\xi w_n s+w_n^2 = \{ S^2+2\xi w_n s+w_n^2 + (\xi w_n)^2 - (\xi w_n)^2 \}$

$= (s+\xi w_n)^2 - w_n^2(\xi^2-1)$

The transfer function becomes,

$$C(s) = \left(\frac{w_n^2}{(s+\xi wn)^2 - wn\text{^}2(-1+\xi\text{^}2)}\right)R(s)$$

Substitute, R(s) = 1/s in the above equation.

$$C(s) = \left(\frac{w_n^2}{(s+\xi wn)^2 - wn\text{^}2(-1+\xi\text{^}2)}\right)\left(\frac{1}{s}\right)$$

$$= \left(\frac{w_n^2}{s\left(s+\xi wn+wn(\xi 2-1)^{1/2}\right)\left(s+\xi wn-wn(\xi 2-1)^{1/2}\right)}\right)$$

Do partial fractions of C(s).

$$C(s) = \left(\frac{w_n^2}{s\left(s+\xi wn+wn(\xi 2-1)^{1/2}\right)\left(s+\xi wn-wn(\xi 2-1)^{1/2}\right)}\right) = \left(\frac{A}{s}\right)+$$
$$\left(\frac{B}{\left(s+\xi wn+wn(\xi 2-1)^{1/2}\right)}\right)+\left(\frac{C}{\left(s+\xi wn-wn(\xi 2-1)^{1/2}\right)}\right)$$

After simplifying, we will get the values of A, B and C as 1, $\frac{1}{2\left(\xi+(\xi 2-1)^{1/2}\right)(\xi 2-1)^{1/2}}$ and $\frac{1}{2\left(\xi-(\xi 2-1)^{1/2}\right)(\xi 2-1)^{1/2}}$ respectively. Substitute these values in above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s}+\left(\frac{1}{2\left(\xi+(\xi 2-1)^{1/2}\right)(\xi 2-1)^{1/2}}\right)\left(\frac{1}{\left(s+\xi wn+wn(\xi 2-1)^{1/2}\right)}\right)-$$
$$\left(\frac{1}{2\left(\xi-(\xi 2-1)^{1/2}\right)(\xi 2-1)^{1/2}}\right)\left(\frac{1}{\left(s+\xi wn-wn(\xi 2-1)^{1/2}\right)}\right)$$

Apply inverse Laplace transform on both the sides.

$$C(t) =\{ 1+\left(\frac{1}{2\left(\xi+(\xi 2-1)^{1/2}\right)(\xi 2-1)^{1/2}}\right)e^{-t(\xi wn + wn(\xi 2 - 1)^{1/2})} -$$
$$\left(\frac{1}{2\left(\xi-(\xi 2-1)^{1/2}\right)(\xi 2-1)^{1/2}}\right) e^{-t(\xi wn - wn(\xi 2 - 1)^{1/2})} \}$$

Since it is over damped, the unit step response of the second order system when $\delta > 1$ will never reach step input in the steady state.

Transient-Response Specifications.

Transient-response characteristics of a control system to a unit-step input, the specifications are-

• **Delay time, td:** The delay time is the time required for the response to reach half the final value the very first time.

• **Rise time, tr:** The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.

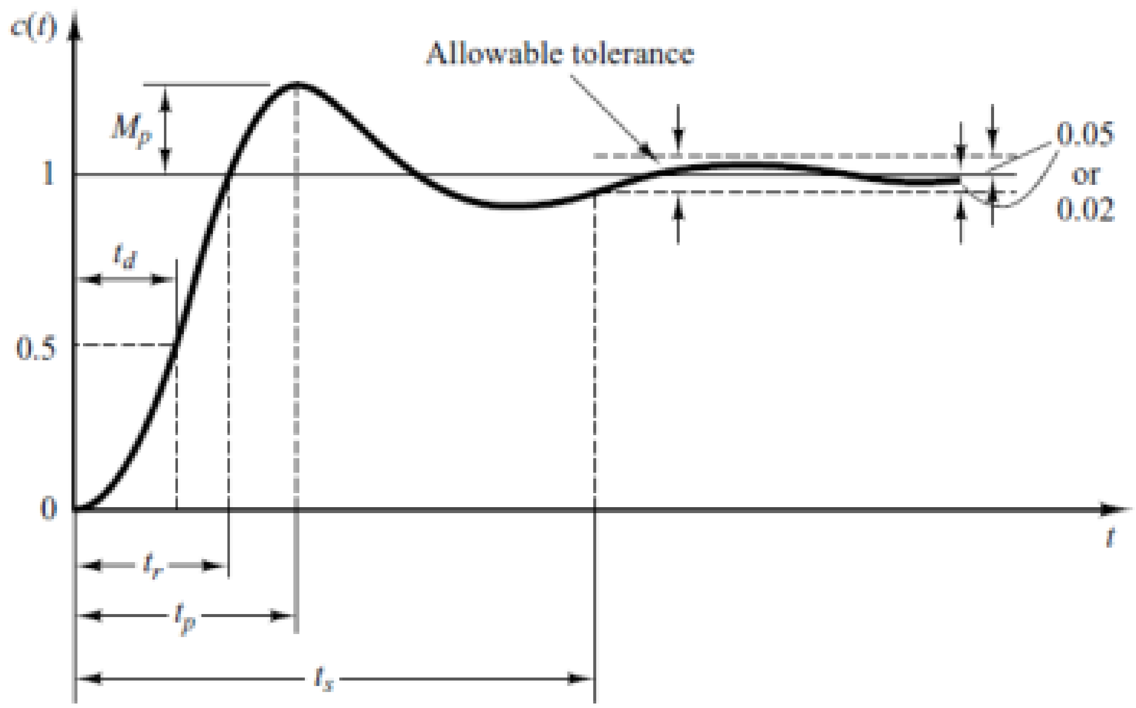
▪ for underdamped second order systems, the 0%to 100%rise time is normally used.

▪ for overdamped systems, the 10% to 90% rise time is commonly used.

• **Peak time, tp:** The peak time is the time required for the response to reach the first peak of the overshoot.

• **Maximum (percent) overshoot, Mp:** The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by-

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$



• **Settling time, ts :** The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system. Which percentage error criterion to use may be determined from the objectives of the system design in question.

Rise time tr : Referring to Eq. c(t), we obtain the rise time tr by letting c(tr)=1.

$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right)$$

$e^{-\zeta\omega_n t_r} \neq 0,$ Since we obtain from Equation the following equation:

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

Since $\omega_d = \omega_n\sqrt{1-\xi^2}$ and $\xi\omega_n = \sigma$, dividing the equation by $\sin\omega_d t_r$

$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

Thus, the rise time t_r is

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi-\beta}{\omega_d}$$

Peak time tp: Referring to Equation, we may obtain the peak time by differentiating c(t) with respect to time and letting this derivative equal zero. Since

$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(\omega_d\sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

and the cosine terms in this last equation cancel each other, dc/dt, evaluated at $t=t_p$, can be simplified to

$$\left.\frac{dc}{dt}\right|_{t=t_p} = (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation:

$$\sin\omega_d t_p = 0$$

Or $\omega_d t_p = 0, \pi, 2\pi, 3\pi, \dots$

Since the peak time corresponds to the first peak overshoot, $\omega_d t_p = \pi.$, Hence

$$t_p = \frac{\pi}{\omega_d}$$

The peak time tp corresponds to one-half cycle of the frequency of damped oscillation.

Maximum overshoot Mp: The maximum overshoot occurs at the peak time or at t=tp=p_vd. Assuming that the final value of the output is unity, Mp is obtained from Equation as

$$M_p = c(t_p) - 1$$
$$= -e^{-\zeta\omega_n(\pi/\omega_d)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$
$$= e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

If the final value c(q) of the output is not unity, then we need to use the following equation:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

Settling time ts : For an underdamped second-order system, the transient response is obtained from Equation as

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0$$

The speed of decay of the transient response depends on the value of the time constant 1/ $\xi\omega n$.

For a given vn, the settling time ts is a function of the damping ratio z The settling time corresponding to a ; 2% or ;5% tolerance band may be measured in terms of the time constant T=1/ $\xi\omega n$. For 0<z<0.9, if the 2% criterion is used, ts is approximately four times the time constant of the system. If the 5% criterion is used, then ts is approximately three times the time constant.

For convenience in comparing the responses of systems, we commonly define the settling time t_s to be

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad (2\% \text{ criterion})$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad (5\% \text{ criterion})$$

Ex.:- when a second order control system is subjected to a unit step input, the values of ξ=0.5 and ωn= 6 rad/sec. determine the rise time, peak time, settling time and peak overshoot.

Soln.:- Rise time is

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi - \tan^{-1}\frac{\sqrt{1-(0.5)^2}}{0.5}}{6\sqrt{1-(0.5)^2}} = \frac{3.14-1.047}{5.19} = 0.403 \text{ sec.}$$

Peak time is

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{6\sqrt{1-(0.5)^2}} = 0.605 \text{ sec.}$$

Settling time is

$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.5\times 6} = 1.33 \text{ sec.}$$

Maximum overshoot is

$$M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}\times 100 = e^{\frac{-\pi\times 0.5}{\sqrt{1-(0.5)^2}}}\times 100$$
$$M_p = 0.163\times 100 = 16.3\%$$

Impulse Response of Second Order System

The impulse response of the second order system can be obtained by using any one of these two methods.

- Follow the procedure involved while deriving step response by considering the value of R(s) as 1 instead of 1/s.
- Do the differentiation of the step response.

The following table shows the impulse response of the second order system for 4 cases of the damping ratio.

Condition of Damping ratio	**Out-put Impulse response C(t), for t>=0**
$\xi=0$	$\omega_n \sin(\omega_n t)$
$\xi=1$	$\omega_n^2 t e^{-\omega_n t}$
$0<\xi 1$	$(\omega_n . t . e^{-\omega_n t}/(1-\xi^2)^{\wedge 1/2}). \sin(\omega_d t)$
$\xi>1$	$\{\omega_n . /2(1-\xi^2)^{\wedge 1/2}\}.\{e^{-[\xi\omega_n - \omega_n(\xi^2-1)^{\wedge 1/2})t} - e^{-[\xi\omega_n + \omega_n(\xi^2-1)^{\wedge 1/2})t}\}$

Unit Ramp Input time Response of Second Order System

The ramp response of the second order system can be obtained by using same steps as step input methods.

- Follow the procedure involved while deriving step response by considering the value of R(s) as $1/s^2$ instead of 1/s.

$$C(s) = \left(\frac{w_n^2}{s^2+2\xi w_n s+w_n^2}\right)R(s)$$

Substitute, $R(s) = 1/s^2$ in the above equation.

$$C(s) = \left(\frac{w_n^2}{s^2+2\xi w_n s+w_n^2}\right).(1/s^2)$$

by using same steps as step input methods than response is as-

$$C(t) = t- \frac{2\xi}{w_n} + \frac{2\xi}{w_n}e^{-}t\xi w_n\cos(w_n\sqrt{1-\xi^2})t + \frac{2\xi 2-1}{w_d}e^{-}t\xi w_n\sin(w_n\sqrt{1-\xi^2})t$$

……………………………………………………

Put $\sin\varnothing=2\xi\sqrt{1-\xi^2}$ and $\cos\varnothing= 2\xi$^$2-1$

$$C(t) = t- \frac{2\xi}{w_n} + \frac{e^{\wedge}-t\xi w_n}{w_d}\sin(w_n\sqrt{1-\xi^2} +\varnothing)$$

Error Signal

e(t) = r(t) - c(t)

$$= t- t- \frac{2\xi}{w_n} + \frac{e^{\wedge}-t\xi w_n}{w_d}\sin(w_n\sqrt{1-\xi^2} +\varnothing)$$

$$= \frac{2\xi}{w_n} - \frac{e^{\wedge}-t\xi w_n}{w_d}\sin(w_n\sqrt{1-\xi^2} +\varnothing)$$

Steady state error is

$e_{ss} = \lim_{t\to\infty} e(t) = 2\xi/w_n$

……..