ERROR ANALYSIS

Classification of control system

Consider the open loop transfer function-

$$G(s)H(s) = \frac{K(1+sT1)(1+sT2)\ldots\ldots}{S^m(1+sTa)(1+sTb)\ldots\ldots} \quad (1)$$

In equation (1), the **poles** are at $s = -1/T_a$, $s = -1/T_b$ ………….and **Zeros** are at $s = -1/T_1$, $s = -1/T_2$ …………. the equation having a term **S^m in denominator**, **'m' is the number of poles at the origin**. A system having no poles at origin of the 's' plane, is said to be **type '0' (zero) system i.e., m = 0**, if **m = 1** i.e., 's' , it means the system has **a pole at origin** of the s-plane and is said to be **type '1' (one) system**. A system is called **type '2' system if m = 2**. And so on.

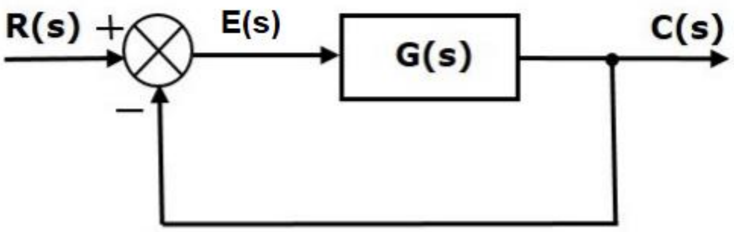
Steady State Error- The steady state error is the difference between the input and output of the system during steady state. For accuracy the steady state error should be minimum.

The deviation of the output of control system from desired response during steady state is known as steady state error. It is represented as e_{ss}. We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \quad (2)$$

Where,
E(s) is the Laplace transform of the error signal, e(t)
Let us discuss how to find steady state errors for unity feedback and non-unity feedback control systems one by one.
Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.



Where,
R(s) is the Laplace transform of the input signal, r(t)
C(s) is the Laplace transform of the output signal, c(t)
We know the transfer function of the unity negative feedback i.e. H(s)=1, closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}$$

The output of the summing point is-

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

Substitute E(s) value in the steady state error formula as in equation (2)

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)} \qquad (3)$$

From the equation (3) it is clear that the steady state error depends on the input and open loop transfer function.

Static Error Coefficients-

(a) Static Position Error Constant Kp-

The steady state error is given by equation (3)

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

Put the value of R(s) in above equation for unit step input R(s) = 1/s, than

$$e_{ss} = \lim_{s \to 0} s.E(s) = \lim_{s \to 0} s.\frac{R(s)}{1+G(s)} = \lim_{s \to 0} s.\frac{1/s}{1+G(s)}$$

$$= \lim_{s \to 0} \frac{1}{1+G(s)} = \frac{1}{1+\lim_{s \to 0} G(s)}$$

$$= \frac{1}{1+Kp} \qquad (4)$$

Kp = static position error constant = $\lim_{s \to 0} G(s).H(s)$ (5)

(b) Static velocity Error Constant Kv-

The steady state error is given by equation (3)

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Put the value of R(s) in above equation for unit ramp input R(s) = $1/s^2$, than

$$e_{ss} = \lim_{s\to 0} s.E(s) = \lim_{s\to 0} s.\frac{R(s)}{1+G(s)} = \lim_{s\to 0} s.\frac{1/s2}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{1}{s(1+G(s))} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \lim_{s\to 0} \frac{1}{sG(s)}$$

$$= \frac{1}{Kv} \qquad (6)$$

K_v = static velocity error constant = $\lim_{s\to 0} s.G(s).H(s)$ (7)

(c) Static acceleration Error Constant Ka-

The steady state error is given by equation (3)

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Put the value of R(s) in above equation for unit parabolic input R(s) = $1/s^3$, than

$$e_{ss} = \lim_{s\to 0} s.E(s) = \lim_{s\to 0} s.\frac{R(s)}{1+G(s)} = \lim_{s\to 0} s.\frac{1/s3}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{1}{s2(1+G(s))} = \lim_{s\to 0} \frac{1}{s2+s2G(s)} = \lim_{s\to 0} \frac{1}{s2G(s)}$$

$$= \frac{1}{Ka} \qquad (8)$$

K_a = static velocity error constant = $\lim_{s\to 0} s^2.G(s).H(s)$ (9)

The following table shows the steady state errors and the error constants for standard input signals like unit step, unit ramp & unit parabolic signals.

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s\to 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s\to 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s\to 0} s^2G(s)$

Where, Kp, Kv and Ka are position error constant, velocity error constant and acceleration error constant respectively.

Note – If any of the above input signals has the amplitude other than unity, then multiply corresponding steady state error with that amplitude.

Note – We can't define the steady state error for the unit impulse signal because, it exists only at origin. So, we can't compare the impulse response with the unit impulse input as t denotes infinity.

Ex.: Find the steady state error for an input signal r(t) = $(5+2t+t^2/2)u(t)$ of unity negative feedback control system with G(s) = $\frac{5(s+4)}{s2(s+1)(s+20)}$.

Soln.:

The given input signal is a combination of three signals- step, ramp and parabolic.

The following table shows the error constants and steady state error values for these three signals.

Input signal	Error constant	Steady state error
$r_1(t) = 5u(t)$	$K_p = \lim_{s\to 0} G(s) = \infty$	$e_{ss1} = \frac{5}{1+k_p} = 0$
$r_2(t) = 2tu(t)$	$K_v = \lim_{s\to 0} sG(s) = \infty$	$e_{ss2} = \frac{2}{K_v} = 0$
$r_3(t) = \frac{t^2}{2}u(t)$	$K_a = \lim_{s\to 0} s^2G(s) = 1$	$e_{ss3} = \frac{1}{k_a} = 1$

We will get the overall steady state error, by adding the above three steady state errors.

$$e_{ss} = e_{ss1}+e_{ss2}+e_{ss3}$$

$$\Rightarrow e_{ss}=0+0+1=1 \Rightarrow e_{ss}=0+0+1=1$$

Therefore, we got the steady state error e_{ss} as 1 for this example.

Steady State Error for Different Type of Systems-

1. **(a) Type zero system with unit step input-**

Consider the open loop transfer function-

$$G(s)H(s) = \frac{K(1+sT1)(1+sT2)\ldots\ldots}{S^m(1+sTa)(1+sTb)\ldots\ldots} \quad (1)$$

Here m=0, put in the above equation

$$G(s)H(s) = \frac{K(1+sT1)(1+sT2)\ldots\ldots}{S^0(1+sTa)(1+sTb)\ldots\ldots} = \frac{K(1+sT1)(1+sT2)\ldots\ldots}{(1+sTa)(1+sTb)\ldots\ldots}$$

As we know the Kp in equation (4) put the value of G(s)H(s)

Kp = static position error constant = $\lim_{s\to 0} \frac{K(1+sT1)(1+sT2)\ldots\ldots}{(1+sTa)(1+sTb)\ldots\ldots} = K$

Steady state error is

ess = 1/(1+Kp) = 1/(1+K)

hence for type zero system the static position error constant Kp is finite.

(b) Type zero system with unit ramp input-

K_v = static velocity error constant = $\lim_{s\to 0} s.\frac{K(1+sT1)(1+sT2)\ldots\ldots}{(1+sTa)(1+sTb)\ldots\ldots} = 0$

$e_{ss} = \frac{1}{Kv} = \frac{1}{0} = \infty$

(c) Type zero system with unit ramp input-

K_a = static velocity error constant = $\lim_{s\to 0} s^2.\frac{K(1+sT1)(1+sT2)\ldots\ldots}{(1+sTa)(1+sTb)\ldots\ldots} = 0$

$e_{ss} = \frac{1}{Ka} = \frac{1}{0} = \infty$

For Type '0' system, the steady state error is infinite for ramp and parabolic inputs. Hence, the ramp and parabolic inputs are not acceptable.

2. **(a) Type '1' system with unit step input (m=1)-**

$G(s)H(s) = \frac{K(1+sT1)(1+sT2).......}{s(1+sTa)(1+sTb).....}$

Kp = $\lim_{s\to 0} G(s).H(s) = \lim_{s\to 0} \frac{K(1+sT1)(1+sT2).......}{s(1+sTa)(1+sTb).....} = \infty$

$e_{ss} = \frac{1}{1+Kp} = \frac{1}{1+\infty} = 0$

(b) Type '1' system with unit ramp input-

Kv = $\lim_{s\to 0} s.\, G(s).H(s) = \lim_{s\to 0} \frac{s.K(1+sT1)(1+sT2).......}{s(1+sTa)(1+sTb).....} = K$

$e_{ss} = \frac{1}{Kv} = \frac{1}{k}$

(c) Type '1' system with unit parabolic input-

Ka = $\lim_{s\to 0} s\,\hat{}2.\, G(s).H(s) = \lim_{s\to 0} \frac{s\hat{}2.K(1+sT1)(1+sT2).......}{s(1+sTa)(1+sTb).....} = 0$

$e_{ss} = \frac{1}{Ka} = \frac{1}{0} = \infty$

Hence, from above relations for type '1' system, it is clear that for type '1' system step input and ramp inputs are acceptable and parabolic input is not acceptable.

3. **(a) Type '2' system with unit step input (m=2)-**

$G(s)H(s) = \frac{K(1+sT1)(1+sT2).......}{s\hat{}2(1+sTa)(1+sTb).....}$

Kp = $\lim_{s\to 0} G(s).H(s) = \lim_{s\to 0} \frac{K(1+sT1)(1+sT2).......}{s\hat{}2(1+sTa)(1+sTb).....} = \infty$

$e_{ss} = \frac{1}{1+Kp} = \frac{1}{1+\infty} = 0$

(b) Type '2' system with unit ramp input-

Kv = $\lim_{s\to 0} s.\, G(s).H(s) = \lim_{s\to 0} \frac{s.K(1+sT1)(1+sT2).......}{s\hat{}2(1+sTa)(1+sTb).....} = \infty$

$e_{ss} = \frac{1}{Kv} = \frac{1}{\infty} = 0$

(c) Type '2' system with unit parabolic input-

Ka = $\lim_{s\to 0} s\,\hat{}2.\, G(s).H(s) = \lim_{s\to 0} \frac{s\hat{}2.K(1+sT1)(1+sT2).......}{s\hat{}2(1+sTa)(1+sTb).....} = K$

$e_{ss} = \frac{1}{Ka} = \frac{1}{K}$

Hence, from above relations for type '2' system, it is clear that for type '2' system step input, ramp inputs and parabolic inputs are acceptable.

Ex.: The forward path transfer function of a unity feedback control system is given by

$G(s) = \frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$, determine the step, ramp and parabolic error coefficients. Also determine the type of the system.

Soln.:

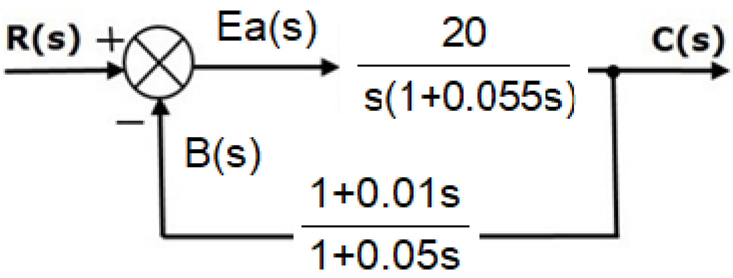
$$\textbf{Kp} = \lim_{s\to 0} G(s).H(s) = \lim_{s\to 0} \frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)} = \infty$$

$$\textbf{Kv} = \lim_{s\to 0} s.\, G(s).H(s) = \lim_{s\to 0} s.\frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)} = \infty$$

$$\textbf{Ka} = \lim_{s\to 0} s^2.\, G(s).H(s) = \lim_{s\to 0} s^2.\frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)} = \frac{5(100)}{(5)(10)} = 10$$

In denominator the value of m = 2. Hence, the given system is type '2' system.

Ex.:- for a given system shown in fig., determine the actuating signal Ea(s). Also determine the position error constant for unit step input.



Soln.:-

$$Ea(s) = 20R(s) - B(s)$$

$$C(s) = \frac{20Ea(s)}{s(1+0.05s)}$$

$$B(s) = C(s)\frac{1+0.01s}{1+0.05s} = \left(\frac{20Ea(s)}{s(1+0.05s)}\right)\left(\frac{1+0.01s}{1+0.05s}\right)$$

$$Ea(s) = 20R(s) - \left(\frac{20Ea(s)}{s(1+0.05s)}\right)\left(\frac{1+0.01s}{1+0.05s}\right)$$

$$Ea(s) = \frac{20s(1+0.05s)(1+0.05s)}{s(1+0.05s)(1+0.05s)+20(1+0.01s)}R(s)$$

For step input, R(s) = 1/s

$$\textbf{e}_{\textbf{ss}} = \lim_{s\to 0} s\, E(s) = \lim_{s\to 0} s\frac{20s(1+0.05s)(1+0.05s)}{s(1+0.05s)(1+0.05s)+20(1+0.01s)}(1/s) = 0$$

$$e_{ss} = \frac{1}{1+Kp} = 0, \text{ than } Kp = \infty$$

Dynamic Error Coefficients-

For the steady state error, the static error coefficients gives the limited information.

The error function is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} \tag{10}$$

For unity feedback system

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} \tag{11}$$

The equation 11 can be expressed in polynomial form as

$$\frac{E(s)}{R(s)} = \frac{1}{K1} S^0 + \frac{1}{K2} S + \frac{1}{K3} S^2 + \ldots\ldots. \tag{12}$$

Or, $E(s) = = \frac{1}{K1} S^0 R(s) + \frac{1}{K2} S^1 R(s) + \frac{1}{K3} S^2 R(s) \ldots\ldots\ldots\ldots$ (13)

Take Inverse Laplace of equation 13, than error is

$e(t) = = \frac{1}{K1} S^0 r(t) + \frac{1}{K2} S^1 r'(t) + \frac{1}{K3} S^2 r''(t) \ldots\ldots\ldots\ldots$ (14)

Steady state error is given by

$$\mathbf{e_{ss}} = \lim_{s\to 0} s\, E(s)$$

Now, steady state error for unit step input i.e. put R(s)=1/s in equation 13 than

$$\mathbf{e_{ss}} = \lim_{s\to 0} s \left\{\frac{1}{K1} S^0 (1/s) + \frac{1}{K2} S^1 (1/s) + \frac{1}{K3} S^2 (1/s) \ldots\ldots\ldots\ldots\right\}$$

$$= 1/K_1$$

Similarly, for other test signal we can find steady state error, K1,K2,K3,………….are known as dynamic error coefficients.

Ex.: Find the dynamic error coefficients of the unity feedback system whose forward path transfer function is G(s) = 200/s(s+5). Find the steady state error to the input $r(t) = 4t^2$.

Solution: Given that $G(s) = \frac{200}{s(s+5)}$, here H(s) = 1.

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = \frac{15s+s^2}{200+5s+s^2}$$

Above equation can be expressed in polynomial form by using Tailor series

$$\frac{E(s)}{R(s)} = 0.025s + 0.004375s^2 - 0.000218s^4 \ldots\ldots\ldots\ldots\ldots\ldots$$

Or, $E(s) = 0.025sR(s) + 0.004375s^2 R(s) - 0.000218s^4 R(s) + \ldots\ldots\ldots\ldots\ldots\ldots$

Take inverse Laplace than

e(t) = 0.025r'(t)+0.004375r''(t)-0.000218r''''(t)+……………..

given that

$r(t) = 4t^2$

r'(t) = 8t

r''(t) = 8

r'''(t) = 0

$\therefore$ e(t) = 0.025*8t+0.004375*8

= 0.2t+0.035

The steady state error is given by

e_{ss} = $\lim_{t\to\infty} e$ (t) = $\lim_{t\to\infty}(0.2t + 0.035)$

The dynamic error coefficients are

K1 = 1/0.025 = 40

K2 = 1/0.004375 = 228.57

K3 = ∞

K4 = 1/0. 0.000218 = 45871.55

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