LAPLACE TRANSFORM

The laplace transform of a function $F(t)$, $t \geq 0$, is defined by

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

It is also denoted by $f(P)$

Function of exponetial order —

A function $F(t)$ is said to be of exponetial order α if there exist constant α and $M > 0$ such that

$$|F(t)| \leq M e^{\alpha t}, \quad t \geq 0$$

Geometrically, this condition implies that the graph of $f(t)$, $t > 0$ does not grow faster than the graph of the exponetial function $M e^{\alpha t}$, $\alpha > 0$.

Since $|t| \leq e^t$, $|e^{-2t}| \leq e^t$

$|\cos t| \leq e^t$, $t \geq 0$ therefore the functions t, e^{-2t} & $\cos t$ are of exponetial order whereas the function e^{ct^2} is not a function of exponetial order.

Sufficient condition for existence of laplace Transform — If $F(t)$ is a piecewise continuous function in $(0, \infty)$ and is of exponetial order α for $t \geq 0$ then laplace transform of $F(t)$ exists for $p > \alpha$.

Laplace Transform of some Elementary functions -

$L\{1\} = \frac{1}{p}$

$L\{e^{at}\} = \frac{1}{p-a}$

$L\{e^{-at}\} = \frac{1}{p+a}$

$L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}}$

$L\{\sin at\} = \frac{a}{p^2+a^2}$

$L\{\cos at\} = \frac{p}{p^2+a^2}$

$L\{\sinh at\} = \frac{a}{p^2-a^2}$

$L\{\cosh at\} = \frac{p}{p^2-a^2}$

Properties of Laplace Transform

1. Linear property -

$L\{c_1 F_1(t) \pm c_2 F_2(t) \pm \ldots\} = c_1 L\{F_1(t)\} \pm c_2 L\{F_2(t)\} \pm \ldots$

2. First shifting Theorem

If $L\{F(t)\} = f(p)$ then $L\{e^{at} F(t)\} = f(p-a)$ and $L\{e^{-at} F(t)\} = f(p+a)$

3. Scale change property

If $L\{F(t)\} = f(p)$ then $L\{F(at)\} = \frac{1}{a} f(p/a)$

we have $L\{F(t)\} = f(p) = \int_0^\infty e^{-tp} F(t)\,dt$ — (1)

$\Rightarrow L\{F(at)\} = \int_0^\infty e^{-tp} F(at)$

put $at = u$

$dt = \frac{1}{a} du$

Now $F\{F(at)\} = \int_0^\infty e^{-pu/a} F(u) \cdot \frac{1}{a} du$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{p}{a}u} F(u)\, du$$

$$= \frac{1}{a} f(p/a) \quad \text{by } (1)$$

(4) Laplace Transform of Derivative

If $L\{F(t)\} = f(p)$ and $F'(t)$ is of exponential order and at least piecewise continuous on $[0,\infty)$ then

$$L\{F'(t)\} = p f(p) - F(0)$$

Similarly $L\{F''(t)\} = p^2 f(p) - p F(0) - F'(0)$

& $L\{F'''(t)\} = p^3 f(p) - p^2 F(0) - p F'(0) - F''(0)$

and, in general,

$$L\{F^{(n)}(t)\} = p^n f(p) - p^{n-1} F(0) - p^{n-2} F'(0) \cdots - F^{(n-1)}(0)$$

(5) Laplace transform of Integral

If $L\{F(t)\} = f(p)$ then

$$L\left\{\int_0^t F(u)\, du\right\} = \frac{f(p)}{p}$$

(6) Laplace transform ~~of~~ of multiplication by t^n

If $L\{F(t)\} = f(p)$ then

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$$

~~we know that~~

$L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\, dt \quad (1)$

so $L\{t^n F(t)$

(7.) **Laplace Transform of division by t**

If $L\{F(t)\} = f(P)$ then

$$L\left\{\frac{F(t)}{t}\right\} = \int_P^{\infty} f(x)\,dx$$

(8) **Second Shifting Theorem**

If $L\{F(t)\} = f(P)$ and

$$G(t) = \begin{cases} F(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$

then $L\{G(t)\} = e^{-aP} f(P)$

OR $L\{F(t-a)\,u(t-a)\} = e^{-aP} f(P)$

where $u(t-a)$ Heaviside's unit step function defined as

$$u(t-a) = \begin{cases} 1, & t > a \\ 0 & t < a \end{cases}$$

(9) **Laplace transform of periodic function**

Let $F(t)$ be a periodic function with period ω i.e. $F(t+\omega) = F(t)$, then

$$L\{F(t)\} = \frac{1}{1 - e^{-P\omega}} \int_0^{\omega} e^{-Pt} F(t)\,dt$$

(10) **Initial value and Final value Theorems**

(i) $\lim_{t \to 0} F(t) = \lim_{P \to \infty} P f(P)$

(ii) $\lim_{t \to \infty} F(t) = \lim_{P \to 0} P f(P)$

where $f(P) = L\{F(t)\}$