Q Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot d\vec{S}$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant.

$$\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot (dy\,dz\,\hat{i} + dz\,dx\,\hat{j} + dx\,dy\,\hat{k})$$

$$= \iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy)$$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} yz\,dz\,dy + \int_0^a \int_0^{\sqrt{a^2-x^2}} zx\,dz\,dx + \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx \quad \text{(1)}$$

$$= \int_0^a y\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy = \int_0^a \frac{y}{2}(a^2-y^2)\,dy$$

$$= \frac{1}{2}\int_0^a (a^2y - y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{a^4}{8}$$

from eq (1) $\longrightarrow$

$$= \frac{a^4}{8} + \frac{a^4}{8} + \frac{a^4}{8}$$

$$= \frac{3a^4}{8} \quad \text{Ans}$$

Gauss Divergence Theorem :-

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv}$$

or

$$\boxed{\int \vec{F}\cdot\hat{n}\,ds = \int \vec{F}\cdot d\vec{s} = \int \nabla\cdot\vec{F}\,dv}$$

Cartesian Form of Gauss Divergence Theorem :-

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

where, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Q Evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$, where S is a closed surface and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

By Gauss Divergence Theorem, we have —

$\int_V \nabla\cdot\vec{r}\,dv = \int 3\,dv = 3V$ Ans

Q For any closed surface S. Evaluate $\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$

By Gauss Divergence Theorem, we have —

$\int_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds = \int_V \operatorname{div}(\operatorname{curl}\vec{F})\,dv$

$= 0$ Ans