17/5/22

Q- find $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$.
and S is the surface of the sphere having center at $(3,-1,2)$ and radius 3.

Solⁿ- By Gauss Divergence theorem we have,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_V \operatorname{div}\vec{F}\cdot dV$$

$$= \int_V \left[\frac{\delta}{\delta x}(2x+3z) - \frac{\delta}{\delta y}(xz+y) + \frac{\delta}{\delta z}(y^2+2z)\right]$$

$$= \int_V (2-1+2)\,dV$$

$$= \int_V 3\,dV$$

$$= 3V = 3\times\frac{4}{3}\pi r^3 = 4\pi r^3$$

$$= 108\pi$$

Q. The vector field $\bar{F} = (x^2\hat{i} + z\hat{j} + yz\hat{k})$ is defined over the volume of the cuboid given by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ enclosing the surface s, evaluate $\iint_S \vec{F} \cdot d\bar{s}$.

Solⁿ- $\int_S \vec{F} \cdot d\bar{s} = \int_V \text{div}\,\vec{F} \cdot dV$

$= \int_V (2x + y)\, dV$

$= \int_0^a \int_0^b \int_0^c (2x + y)\, dz\, dy\, dx$

$= \int_0^a \int_0^b (2x+y)\, c \; dy\, dx$

$= c \int_0^a \left[2xy + \left\{ \frac{y^2}{2} \right\} \right]_0^b dx$

$= c \int_0^a \left(2xb + \frac{b^2}{2} \right) dx$

$= bc \left\{ \left[x^2 + \frac{bx}{2} \right]_0^a \right.$

$= bc \left(a^2 + \frac{ab}{2} \right)$

$= a^2bc + \frac{ab^2c}{2}.$

S. Evaluate $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$

where S is the portion of the plane $x + 2y + 3z = 6$ which lies in the first octant.

Ans:
$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint \nabla\cdot\vec{F}\,dV$$

$$\vec{F}\cdot\hat{n}\,ds = (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy).$$

$$\hat{n}\,ds = \overline{ds}$$
$$= \left(\hat{i}\,dy\,dz + \hat{j}\,dx\,dz + \hat{k}\,dx\,dy\right)$$

$$\Rightarrow \vec{F} = (x\hat{i} + y\hat{j} + z\hat{k})$$
$$\nabla\cdot\vec{F} = 3$$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V 3\,dV$$

$$\cancel{= 3\iiint dz\,dy\,d}$$

$$= 3\int_0^6 \int_0^{\frac{6-x}{2}} \int_0^{\frac{6-x-2y}{3}} dz\,dy\,dx$$

$$= 3\int_0^6 \int_0^{\frac{6-x}{2}} \left[\frac{6-x-2y}{3}\right] dy\,dx.$$

$$= \int_0^6 \left[6y - xy - \frac{\cancel{2}y^2}{\cancel{2}}\right]_0^{\frac{6-x}{2}} dx$$

$$= \int_0^6 \left[3(6-x) - \frac{x(6-x)}{2} - \left(\frac{6-x}{2}\right)^2\right] dx$$

$$= \int_0^6 \left[18 - \cancel{3x} - \frac{6x}{2} + \frac{x^2}{2} - \frac{36}{4} - \frac{x^2}{4} + \cancel{3x}\right] dx$$

$$= \int_0^6 \left[9 - 3x + \frac{x^2}{4} \right] dx$$

$$= \left[9x - \frac{3x^2}{2} + \frac{x^3}{12} \right]_0^6$$

$$= \left(54 - 54 + 18 \right)$$

$$= 18$$