Q. Prove That $\rightarrow \text{div}(\text{grad}\,\phi) = \nabla^2\phi$

$$\text{div}(\text{grad}\,\phi) = \nabla\cdot(\nabla\phi)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\text{div}(\text{grad}\,\phi) = \nabla^2\phi$$

—— Proved

Q. Prove That $\rightarrow \text{curl}(\text{grad}\,\phi) = 0$

$$\text{curl}(\text{grad}\,\phi) = \nabla\times(\text{grad}\,\phi)$$

$$= \nabla\times\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$\text{curl}(\text{grad}\,\phi) = \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2 y}{\partial y\partial z}\right) - \hat{j}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$\text{curl}(\text{grad}\,\phi) = 0$$

—— Proved

Q. Prove That $\rightarrow$ $\text{div}(\text{curl}\,\vec{V}) = 0$

let $\vec{V} = \hat{i}V_1 + \hat{j}V_2 + \hat{k}V_3$

$$\text{curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$\text{curl}\,\vec{V} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\text{div}(\text{curl}\,\vec{V}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left\{\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right\}$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_3}{\partial x \partial y} + \frac{\partial^2 V_1}{\partial y \partial z} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial y \partial z}$$

$$\text{div}(\text{curl}\,\vec{V}) = 0$$

——— Proved

Q. Prove That $\rightarrow$ $\text{div}(u\vec{a}) = u\,\text{div}\,\vec{a} + (\text{grad}\,u)\cdot\vec{a}$

$$\text{div}(u\vec{a}) = \sum \hat{i}\cdot\frac{\partial}{\partial x}(u\vec{a})$$

$$= \sum \hat{i}\left(u\frac{\partial \vec{a}}{\partial x} + \frac{\partial u}{\partial x}\cdot\vec{a}\right)$$

$$= u\sum \hat{i}\frac{\partial \vec{a}}{\partial x} + \sum\left(\hat{i}\frac{\partial u}{\partial x}\right)\cdot\vec{a}$$

$$\text{div}(u\vec{a}) = u\,\text{div}\,\vec{a} + (\text{grad}\,u)\cdot\vec{a}$$

——— Proved

H.W.

Q. Prove That → $\text{curl}(u\vec{a}) = u\,\text{curl}\,\vec{a} + (\text{grad}\,u) \times \vec{a}$

Q. Prove That → $\text{div}(\text{grad}\,r^n) = \nabla^2(r^n) = n(n+1)\,r^{n-2}$
where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
Hence show that → $\nabla^2\left(\frac{1}{r}\right) = 0$

$$
\begin{aligned}
\text{div}(\text{grad}\,r^n) &= \text{div}(n r^{n-2}\cdot\vec{r}) \\
&= n r^{n-2}\,\text{div}\,\vec{r} + \text{grad}(n r^{n-2})\cdot\vec{r} \\
&= 3n r^{n-2} + \text{grad}(n r^{n-2})\cdot\vec{r} \\
&= 3n r^{n-2} + n(n-2)\,r^{n-3}\,\hat{r}\cdot\vec{r} \\
&= 3n r^{n-2} + n(n-2)\,r^{n-4}\,\vec{r}\cdot\vec{r} \\
&= 3n r^{n-2} + n(n-2)\,r^{n-4}\,r^2 \\
&= 3n r^{n-2} + n(n-2)\,r^{n-2}
\end{aligned}
$$

$$\text{div}(\text{grad}\,r^n) = n(n+1)\,r^{n-2}$$

$$\text{div}(\text{grad}\,r^n) = \nabla^2(r^n) = n(n+1)\,r^{n-2}$$

—— Proved

Put $n = -1$

$$\nabla^2\left(\frac{1}{r}\right) = 0$$

—— Proved