Q. Prove That → the vector $f(r)\vec{r}$ is irrotational.

We have to show that → $\text{curl}\{f(r)\vec{r}\} = 0$

$$\text{curl}\{f(r)\vec{r}\} = f(r)(\text{curl}\,\vec{r}) + \{\text{grad}\, f(r)\} \times \vec{r}$$

$$\text{curl}\,\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{curl}\,\vec{r} = 0$$

$$\text{curl}\{f(r)\vec{r}\} = 0 + f'(r)\cdot\hat{r} \times \vec{r}$$

$$= f'(r)\frac{\vec{r}}{r} \times \vec{r}$$

$$= \frac{f'(r)}{r}(\vec{r} \times \vec{r})$$

$$\text{curl}\{f(r)\vec{r}\} = 0$$

Hence the vector $f(r)\vec{r}$ is irrotational.

—— Proved

Q. Prove That → $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

Hence evaluate $\nabla^2 \log r$ if $r = (x^2+y^2+z^2)^{1/2}$

$$\nabla^2 f(r) = \text{div}\{\text{grad}\, f(r)\}$$
$$= \text{div}\{f'(r)\,\hat{r}\}$$
$$= \text{div}\left\{\frac{f'(r)\,\vec{r}}{r}\right\}$$
$$= \frac{f'(r)}{r}\,\text{div}\,\vec{r} + \text{grad}\left\{\frac{f'(r)}{r}\right\}\cdot\vec{r}$$
$$= \frac{3f'(r)}{r} + \left\{\frac{rf''(r) - f'(r)}{r^2}\right\}\hat{r}\cdot\vec{r}$$
$$= \frac{3f'(r)}{r} + \left\{\frac{rf''(r) - f'(r)}{r^2}\right\}\frac{\vec{r}\cdot\vec{r}}{r}$$
$$= \frac{3f'(r)}{r} + \left\{\frac{rf''(r) - f'(r)}{r^3}\right\}r^2$$
$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$
$$\nabla^2 f(r) = f''(r) + \frac{2f'(r)}{r}$$

—— Proved

Put $f(r) = \log r$

$$\nabla^2 \log r = \frac{-1}{r^2} + \frac{2}{r^2}$$
$$= \frac{1}{r^2}$$
$$\nabla^2 \log r = \frac{1}{x^2+y^2+z^2}$$

Ans