Matrix Eigenvalue Problem -

Let A be a square matrix of order n with elements a_{ij}. Now, we want to find out the column vector X and a constant λ such that.

$$AX = \lambda X \quad \text{—(1)}$$

where λ is called the eigenvalue and X is called the corresponding vector.

The matrix eq. (1) represents a set of homogeneous linear equations.

$$\left.\begin{aligned}(a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0\\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0\\ &\vdots\\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0\end{aligned}\right\} \text{—(2)}$$

A nontrivial solution of these equations exist only when the coefficient determinant in eq. (2) vanishes. We have

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \text{—(3)}$$

This equation is called the characteristic equation of the matrix A is a polynomial equation of degree n in λ, the polynomial being called the characteristic polynomial of A. If the roots of eq. (3) be given by λ_i $(i = 1, 2, \ldots, n)$ then for value of λ_i, there exists a corresponding X_i such that

$$AX_i = \lambda_i X_i \quad \text{—(4)}$$

The eigenvalues λ_i may be either distict or repeated.

the set of all eigenvalues, λ_i of a matrix A is called the spectrum of A and the largest of $|\lambda_i|$ is called the spectral radius of A.

Ex-(A). Find the eigenvalue and eigenvectors of the matrix.

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution. the characteristic eq. of this matrix is given by.

$$\begin{vmatrix} 5-\lambda & 0 & 1 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{vmatrix} = 0$$

we get by $\lambda_1 = -2, \lambda_2 = 4, \lambda_3 = 6$
solving characteristic equation.

Now we are solving for the eigenvectors.

i) $\lambda_1 = -2$ let the eigenvector be

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$7x_1 + x_3 = 0$$

$$x_1 + 7x_3 = 0$$

$$x_1 = 0$$

$$x_3 = 0$$

$x_2 =$ arbitrary

If we take $x_2 = 1$

the eigenvector is $X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(ii) $\lambda_2 = 4$ $\quad X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -6 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

as the eigenvector, the equations are,

$$x_1 + x_3 = 0$$
$$-6x_2 = 0$$
$$x_1 = -x_3$$
$$x_2 = 0$$

$X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ (By normalization)

$$x_1^2 + x_2^2 + x_3^2 = 1$$
$$\frac{1}{2} + 0 + \frac{1}{2} = 1$$

(satisfied)

(iii) $\lambda_3 = 6$ $\quad$ if $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

matrix A is.

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$-x_1 + x_3 = 0$$
$$-8x_2 = 0$$
$$x_1 - x_3 = 0$$

this gives.

$$x_1 = x_3$$
$$x_2 = 0$$

$X_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

this is the normalized eigenvector.

Power method for calculating approximate eigen values-

the eigenvalues of an $n \times n$ matrix A are obtained by solving its characteristic equation,

$$\lambda^n + C_{n-1}\lambda^{n-1} + C_{n-2}\lambda^{n-2} + \cdots + C_0 = 0$$

the method can be used only to find out the eigenvalue of A that is largest in absolute value, which we call dominant eigenvalue of A.

Dominant Eigenvalue and Dominant Eigenvector-

If $\lambda_1, \lambda_2, \cdots$ and λ_n be the eigenvalues of an $n \times n$ matrix A. λ_1 is called the dominant eigenvalue of A if.

$$|\lambda_1| > |\lambda_i|, \quad i = 1, 2, \cdots, n$$

the corresponding to λ_1 are called dominant eigenvectors of A.

* Every matrix has not a dominant eigen value.

Ex-(b). Find the dominant eigenvalue and corresponding eigenvector of the matrix,

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Solution. Characteristic polynomial of A is

$$\begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix}$$

$$-(2-\lambda)(5+\lambda) + 12 = 0$$

$$-(10 + 2\lambda - 5\lambda - \lambda^2) + 12 = 0$$

$$(\lambda+1)(\lambda+2) \Rightarrow \lambda_1 = -1$$

$$\lambda_2 = -2$$

the dominant eigenvalue is $\lambda_2 = -2$.
Now calculate the dominant eigenvector of A is.

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} - 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$4x_1 - 12x_3 = 0$$

$$x_1 - 3x_3 = 0$$

$$x_1 = 3x_3$$

Suppose x_3 has t value

$x_1 = 3t$

$$x = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0$$

the Power Method - like Jacobi method, the power method for approximating eigenvalues is iterative. First we assume that the matrix A has a dominant eigenvalue with corresponding dominant eigenvectors. Then we choose an initial approximation x_0 of one of the dominant eigenvectors of A. the initial approximation must be a nonzero vector in R^n. Finally we make the sequence given by.

$$x_1 = Ax_0$$

$$x_2 = Ax_1 = A(Ax_0) = A^2x_0$$

$$x_3 = Ax_2 = A(A^2x_0) = A^3x_0$$

$$\vdots$$

$$x_k = Ax_{k-1} = A(A^{k-1}x_0) = A^kx_0$$

For large powers of k, and by properly scaling this sequence, we obtain a good approximation of the dominant eigenvector of A.