① Trapezoidal Rule → this is the numerical integration method.

Numerical Integration → Given a set of data points $(x_0, y_0), (x_1, y_1) \cdots (x_n, y_n)$ of a function $y = f(x)$, where $f(x)$ is not known explicitly, it is required to compute the value of definite integral.

$$I = \int_a^b y \, dx \quad —①$$

we derive in this section a general formula for numerical integration using Newton's forward difference formula.

let the interval $[a, b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < x_2 \cdots < x_n = b$

clearly $x_n = x_0 + nh$. Hence the integral becomes.

$$I = \int_{x_0}^{x_n} y \, dx \quad —②$$

Now approximate y by Newton's forward difference formula we obtain,

$$I = \int_{x_0}^{x_n} \left[y_0 + p \Delta y_0 + \frac{p(p+1)}{2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{6} \Delta^3 y_0 + \cdots \right] dx$$

Since $x = x_0 + ph$, $dx = h \, dp$ so the integral becomes.

$$I = h \int_0^n \left[y_0 + p \Delta y_0 + \frac{p(p-1)}{2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{6} \Delta^3 y_0 + \cdots \right] dp$$

By simplifying this equation,

$$\int_{x_0}^{x_n} y \, dx = nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \cdots \right] \quad —③$$

From this general formula, we can obtain different integration formulae by putting $n = 1, 2, 3, \ldots$ etc.

Now we come to trapezoidal Rule.

Put $n=1$ in eq. (3)

$$\int_{x_0}^{x_1} y\, dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{h}{2}(y_0 + y_1) \quad \text{--- (4)}$$

for next interval $[x_1, x_2]$ we deduce similarly,

$$\int_{x_1}^{x_2} y\, dx = \frac{h}{2}(y_1 + y_2) \quad \text{--- (5)}$$

For the last interval $[x_{n-1}, x_n]$ we have.

$$\int_{x_{n-1}}^{x_n} y\, dx = \frac{h}{2}\left[y_{n-1} + y_n\right] \quad \text{--- (6)}$$

Now combining all expressions, we find the rule.

$$\boxed{\int_{x_0}^{x_n} y\, dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]} \quad \text{--- (7)}$$

Equation (7) is known as trapezoidal Rule.

the geometrical significance of this rule is that the curve $y = f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1), (x_1, y_1) and (x_2, y_2) --- (x_{n-1}, y_{n-1}) and (x_n, y_n)

the area bounded by the curve $y=f(x)$, the ordinates $x=x_0$ and $x=x_n$ and the x-axis is then approximately equivalent to the sum of the areas of the n trapeziums obtained.

Error estimation of the trapezoidal formula-

Let $y=f(x)$ be continuous, well-behaved and posses continuous derivatives in $[x_0, x_n]$. Now Expanding y in a Taylor's series around $x=x_0$.

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1} \left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots\right] dx$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{6}y_0''' + \cdots \quad (1)$$

Similarly,

$$\frac{h}{2}(y_0+y_1) = \frac{h}{2}\left(y_0+y_0+hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \cdots\right)$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{4}y_0'' \quad (2)$$

From eq. (1) and (2),

$$\int_{x_0}^{x_1} y\,dx - \frac{h}{2}(y_0+y_1) = -\frac{1}{12}h^3 y_0'' + \cdots \quad (3)$$

this is the error in the interval $[x_0, x_1]$

$$E = -\frac{1}{12}h^3\left(y_0'' + y_1'' + \cdots + y_{n-1}''\right) \quad (4)$$

where E is the total error.

Newton-Cotes Integration formulae → Let the interpolation points x_i, be equally spaced, i.e., let $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots n$, and let the end points of the interval of integration be placed such that $x_0 = a$, $x_n = b$, $h = \frac{b-a}{n}$

then the definite integral,

$$I = \int_a^b y\,dx \quad -(1)$$

is evaluated by an integration formula of the type.

$$I_n = \sum_{i=0}^{n} C_i y_i \quad -(2)$$

eq.(2) is known as Newton-cotes closed integration formulae. they are closed since the end points a and b are the extreme abscissae in the formulae.

On the ~~end~~ other hand, formulae which do not employ the end points are called Newton-cotes open integration formula. Here are given below the five simplest Newton-cotes open integration formulae.

(a) $$\int_{x_0}^{x_2} y\,dx = 2hy_1 + \frac{h^3}{3} y''(\bar{x}) ; (x_0 < \bar{x} < x_2)$$

(b) $$\int_{x_0}^{x_3} y\,dx = \frac{3h}{2}(y_1 + y_2) + \frac{3h^3}{4} y''(\bar{x}) ; (x_0 < \bar{x} < x_3)$$

(c) $$\int_{x_0}^{x_4} y\,dx = \frac{4h}{3}(2y_1 - y_2 + 2y_3) + \frac{14}{45} h^5 y^{iv}(\bar{x})$$

$$(x_0 \le \bar{x} < x_4)$$

(d) $$\int_{x_0}^{x_5} y\,dx = \frac{5h}{24}(11y_1 + y_2 + y_3 + 11y_4) + \frac{95}{144} h^5 y^{iv}(\bar{x}), \quad (x_0 < \bar{x} < x_5)$$