Numerical Solution of Partial Differential Equation

the general second order linear partial differential equation is of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

which can be written as,

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad -①$$

where A, B, C, D, E, F and G are all functions of x and y.

Equations of the form ① can be classified with respect to the sign of the discriminent

$$\Delta = B^2 - 4AC \quad -②$$

where Δ is computed at any point in the (x,y) plane. eq. ① is said to be elliptic, parabolic or hyperbolic as $\Delta < 0$, $\Delta = 0$ or $\Delta > 0$.

For Ex: $u_{xx} + u_{yy} = 0$ (laplace Equation) is elliptic $-③$

$u_{xx} - u_{yy} = 0$ (Wave Equation) is hyperbolic $-④$

$u_t = u_{xx}$ (heat conduction equation) is parabolic $-⑤$

In the study of partial differential equations, usually three types of problems arise:-

① Dirichlet's Problem – Given a continuous function f on the boundary C of a region R, it is required to find a function u(x,y) satisfying the Laplace equation in R i.e., to find $u(x,y)$ such that,

$$\left.\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{in } R, \\ \text{and} \quad u &= f \quad \text{on } C \end{aligned}\right\} \quad -(6)$$

(ii) Cauchy's Problem →

$$\left.\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{for } t>0 \\ u(x,0) &= f(x) \\ \text{and} \quad \frac{\partial u(x,0)}{\partial t} &= g(x) \end{aligned}\right\} \quad -(7)$$

where $f(x)$ and $g(x)$ are arbitrary.

(iii)

$$\left.\begin{aligned} u_t &= u_{xx} \quad \text{for } t>0 \\ \text{and} \quad u(x,0) &= f(x) \end{aligned}\right\} \quad -(8)$$

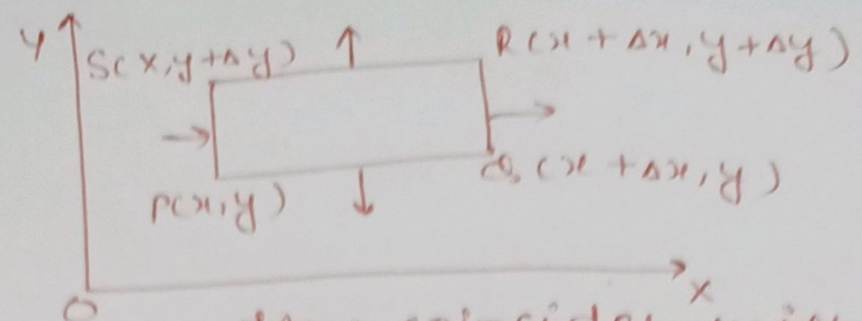
In partial differential equations, the form of the equation is always associated with a particular type of boundary conditions. In this case, the problem is said to be well-defined (well-posed).

The problem defined by.

$$\left.\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0) &= f(x) \\ \text{and} \quad u_y(x,0) &= g(x) \end{aligned}\right\} \quad -(9)$$

Laplace eq. with Cauchy boundary conditions the problem is said to be ill-posed.

Laplace's Equation →



x-y plane coincides with a rectangular face PQRS.

the eq. defined by - $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ — (10)

is called Laplace's equation.

to derive this eq. we consider a heated plate which is insulated everywhere except at its edges where the temp. is constant. Assuming that the xy-plane coincides with one rectangular face PQRS.

the quantity of heat entering the face PS in time Δt = $-k\alpha \Delta y \left[\frac{\partial u}{\partial x}\right]_x \Delta t$,

where α is thickness of the plate.

$u(x, t)$ is the temperature at a distance x at time t and k is the conductivity of the material of the plate.

the amount of heat leaving the face QR in time Δt is $-k\alpha \Delta y \left[\frac{\partial u}{\partial x}\right]_{x+\Delta x} \Delta t$.

By above two expressions, we obtain the gain of heat during time Δt.

$$= k\alpha \Delta y \left[\left[\frac{\partial u}{\partial x}\right]_{x+\Delta x} - \left[\frac{\partial u}{\partial x}\right]_x \right] \Delta t$$

In same manner, we obtain the gain of heat from the faces PQ and SR in time Δt as.

$$k \alpha \Delta x \left[\left(\frac{\partial u}{\partial x} \right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y} \right)_{y} \right] \Delta t.$$

Hence the total gain of heat in the plate.

$$\Rightarrow k \alpha \Delta x \Delta y \left[\frac{\left(\frac{\partial u}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x} \right)_{x}}{\Delta x} + \frac{\left(\frac{\partial u}{\partial y} \right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y} \right)_{y}}{\Delta y} \right]$$

this heat raises the temp. in the plate equals Δt to.

$$\rho \alpha s \Delta x \Delta y \Delta u$$

where s is specific heat
ρ is density of material