Partial Differentiation

Functions of Several Independent Variables

Let u be a symbol which has one definite value for every pair of values of x and y. Then u is called a function of the two independent variables x and y and is written as $u = f(x, y)$.

Partial differential coefficients

Given a function of two variables, $f(x, y)$, the derivative with respect to x only (treating y as a constant) is called the partial derivative of f with respect to x and is denoted by either $\partial f / \partial x$ or f_x.

Similarly, the derivative of f with respect to y only (treating x as a constant) is called the partial derivative of f with respect to y and is denoted by either

$\partial f / \partial y$ or f_y.

- Partial Differentiate f with respect to x twice. (That is, partial differentiate f with respect to x; then partial differentiate the result with respect to x again.)

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}$$

- Partial Differentiate f with respect to y twice. (That is, partial differentiate f with respect to y; then partial differentiate the result with respect to y again.)

$$\frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}$$

Mixed partials:

- First partial differentiate f with respect to x; then partial differentiate the result with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x}$$

- First partial differentiate f with respect to y; then partial differentiate the result with respect to x.

$$\frac{\partial^2 f}{\partial x \partial y}$$

Example 1: Determine the partial derivative of the function: f (x, y) = 3x + 4y.

Solution:

Given function: f (x,y) = 3x + 4y

To find $\partial f/\partial x$, keep y as constant and differentiate the function:

Therefore, $\partial f/\partial x = 3$

Similarly, to find $\partial f/\partial y$, keep x as constant and differentiate the function:

Therefore, $\partial f/\partial y = 4$

Example If $u = ax^2 + 2hxy + by^2$, find $\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x \partial y}$.

Solution we have

$$u = ax^2 + 2hxy + by^2$$

$$\frac{\partial u}{\partial x} = 2ax + 2hy$$

$$\frac{\partial^2 u}{\partial y \partial x} = 2h$$

$$\frac{\partial u}{\partial y} = 2hx + 2by$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2h$$

Example If

$$u = e^{xyz},$$
$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}(1+3xyz+x^2y^2z^2)$$

Solution we have

$$u = e^{xyz}$$

$$\frac{\partial u}{\partial z} = e^{xyz}.xy$$

$$\frac{\partial^2 u}{\partial y\, \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(xye^{xyz}\right) = xe^{xyz} + x^2\, yz e^{xyz}$$

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\, \partial z}\right) = \frac{\partial}{\partial x}\left(xe^{xyz} + x^2\, yz e^{xyz}\right)$$

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial x}\left(xe^{xyz} + x^2\, yz e^{xyz}\right) = e^{xyz} + xyz\, e^{xyz} + 2xyz e^{xyz} + x^2\, yz e^{xyz} yz$$

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = e^{xyz}\{1 + 3xyz + x^2\, y^2\, z^2\}$$

Example If

$$z = \left(\frac{x^2 + y^2}{x + y}\right),$$
$$show\ that \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y})$$

Solution we have

$$z = \left(\frac{x^2 + y^2}{x + y}\right)$$

$$\frac{\partial z}{\partial x}=\left(\frac{(x+y)(2x)-(x^2+y^2)(1)}{(x+y)^2}\right)$$

$$\frac{\partial z}{\partial x}=\left(\frac{x^2-y^2+2xy}{(x+y)^2}\right)$$

$$and \quad \frac{\partial z}{\partial y}=\left(\frac{(x+y)(2y)-(x^2+y^2)(1)}{(x+y)^2}\right)$$

$$\frac{\partial z}{\partial y}=\left(\frac{y^2-x^2+2xy}{(x+y)^2}\right)$$

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=\left\{\frac{(x^2-y^2+2xy)-(y^2-x^2+2xy)}{(x+y)^2}\right\}^2$$

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=\left\{\frac{2(x^2-y^2)}{(x+y)^2}\right\}^2=4\left(\frac{x-y}{x+y}\right)^2 \qquad (1)$$

$$(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y})=1-\left(\frac{x^2-y^2+2xy}{(x+y)^2}\right)-\left(\frac{y^2-x^2+2xy}{(x+y)^2}\right)$$

$$(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y})=\frac{(x+y)^2-x^2+y^2-2xy-y^2+x^2-2xy}{(x+y)^2}=\frac{x^2+y^2+2xy-x^2+y^2-2xy-y^2+x^2-2xy}{(x+y)^2}$$

$$(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y})=\left(\frac{y^2+x^2-2xy}{(x+y)^2}\right)=\frac{(x-y)^2}{(x+y)^2} \qquad (2)$$

From (1) and (2)

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y})$$

Example If

$$u = \log\ (x^3 + y^3 + z^3),$$
$$show\ that\ \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{3\left(x^2 + y^2 + z^2\right)}{x^3 + y^3 + z^3}$$

Solution we have

$$u = \log\ (x^3 + y^3 + z^3)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3}(3x^2)$$

$$\frac{\partial u}{\partial x} = \frac{3(x^2)}{x^3 + y^3 + z^3}$$

$$\frac{\partial u}{\partial y} = \frac{3(y^2)}{x^3 + y^3 + z^3}$$

$$\frac{\partial u}{\partial z} = \frac{3(z^2)}{x^3 + y^3 + z^3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2)}{x^3 + y^3 + z^3}$$

Example If

$$u = \log\ (x^3 + y^3 + z^3 - 3xyz),$$
$$show\ that\ \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{3}{x + y + z}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{\left(x + y + z\right)^2}$$

Solution we have

$$u = \log\ (x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$$

$$\frac{\partial u}{\partial x}=\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y}=\frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z}=\frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}=\frac{3}{(x+y+z)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\frac{3}{(x+y+z)}$$

$$=\left(\frac{\partial}{\partial x}\frac{3}{(x+y+z)}+\frac{\partial}{\partial y}\frac{3}{(x+y+z)}+\frac{\partial}{\partial z}\frac{3}{(x+y+z)}\right)$$

$$=3\left(-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right)$$

$$=-9\left(\frac{1}{(x+y+z)^2}\right)$$

ExampleIf $x=r\cos\theta,\ y=r\sin\theta,$ *prove that*

$$(i)\ \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$$

$$(i)\ \frac{\partial^2 r}{\partial x^2}.\frac{\partial^2 r}{\partial y^2}=\left(\frac{\partial^2 r}{\partial x\partial y}\right)^2$$

$$(iii)\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2=1$$

Solution

$x = r\cos\theta,\ y = r\sin\theta$

Then

$x^2 + y^2 = r^2$

Differentiating Partially w.r. to x and y , we get

$$2x + 0 = 2r\frac{\partial r}{\partial x} \text{ and } 2y + 0 = 2r\frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r}$$

Again Differentiating Partially w.r. to x and y, we get

$$\frac{\partial^2 r}{\partial x^2} = \frac{r.1 - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x\frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3}$$

$$= \frac{(x^2 + y^2) - x^2}{r^3} = \frac{y^2}{r^3}$$

Similarly $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3} + \frac{x^2}{r^3} = \frac{x^2 + y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r}\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] == \frac{r^2}{r^3} = \frac{1}{r}$$

$$\frac{\partial^2 r}{\partial^2 x} + \frac{\partial^2 r}{\partial^2 y} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

(ii) Differentiating $\frac{\partial r}{\partial y}$ partially w.r.t. x, we get

$$\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial r}{\partial x} \frac{y}{r} = \frac{-y}{r^2} \frac{\partial r}{\partial x} = \frac{-xy}{r^3}$$

$$\frac{\partial^2 r}{\partial^2 x} \cdot \frac{\partial^2 r}{\partial^2 y} = \frac{x^2}{r^3} \cdot \frac{y^2}{r^3} = \frac{x^2 y^2}{r^6} = \left(\frac{-xy}{r^3} \right)^2 = \left(\frac{\partial^2 r}{\partial x \partial y} \right)^2$$

(iii)

We have

$$\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$