Poisson's Equation $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ — ①

Laplace's Equation $\nabla^2 V = 0$ — ②

Laplace's eq. follows Poisson's eq. in the region where charge density $\rho = 0$.

Solutions of Laplace's eq. are harmonic functions and have no local maxima or minima.

Earnshaw's theorem → A charged particle cannot be held in stable equilibrium by electrostatic forces alone. This basically follows from the fact that Laplace's equation tolerates no local maxima or minima of the potential and Poisson's eq. allows no stability.

⊙ Stable equilibrium demands extrema of the potential V and i.e. $\nabla^2 V \gtrless 0$. But in the region without any charge density $\nabla^2 V = 0$

⊙ For an assemble of positive charge $\rho > 0$ to be stable it must be at minimum of potential $\nabla^2 V > 0$

But Poisson's equation $\nabla^2 V = -\rho/\epsilon_0 < 0$ gives negative sign indicating maximum of V.

⊙ For an assemble of negative charge $\rho < 0$ to be stable, it must be at maximum of potential i.e., $\nabla^2 V < 0$. But Poisson's equation $\nabla^2 V = -\rho/\epsilon_0 > 0$ gives positive sign indicating minimum of V.

These imply there can be no points of stable equilibrium in an electrostatic field.

Ex-① for Spherical Symmetric Problem-

Apply laplace's equation,

$$\nabla^2 V = 0$$

∇^2 in Spherical coordinates :-

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Here we are considering that the potential function to $V = V(\theta)$.

So laplace eq. becomes-

$$\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

$$\int\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = \int 0$$

$$\sin\theta\frac{dV}{d\theta} = \text{constant}$$

$$\sin\theta\frac{dV}{d\theta} = A$$

$$\frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

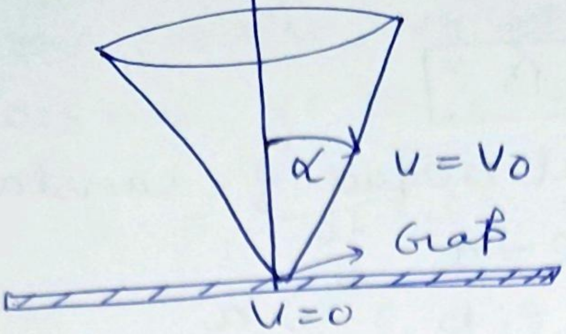
$$\int dV = \int\frac{A}{\sin\theta}\,d\theta + 1$$

$$\boxed{V = \int\frac{A\,d\theta}{\sin\theta} + B}$$

Solution of this equation-

$$V = A\ln\left(\tan\frac{\theta}{2}\right) + B$$

Here the equipotential surfaces are cones.



$\theta = \alpha$, $\alpha < \frac{\pi}{2}$
then $V = V_0$

$\theta = \frac{\pi}{2}$, $V = 0$

we obtain $\quad 0 = A\ln\left(\tan\frac{\pi}{4}\right) + B$

$$B = A\ln\left(\tan\frac{\pi}{4}\right)$$

$$B = A\ln 1$$

$$B = 0$$

$$V_0 = A\ln\left(\tan\frac{\alpha}{2}\right) + B$$

$$\boxed{V = V_0 \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)}}$$

<u>Ex-②</u> <u>For Cylindrical coordinates problem-</u>

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

laplace's equation $\nabla^2 V = 0$

here we are considering the potential V is only the function of ρ

$$\nabla^2 V(\rho) = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0$$

$$\rho \neq 0$$

$$\int\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = \int 0$$

$$\rho\frac{dV}{d\rho} = A$$

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$$\frac{dV}{d\rho} = \frac{A}{\rho}$$

$$\boxed{V = A \ln \rho + B}$$

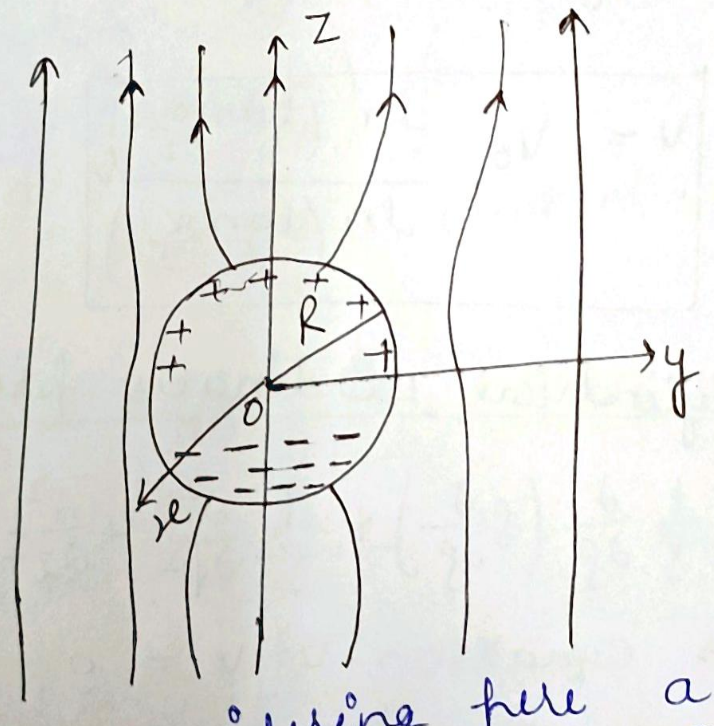
for the equipotential surface ρ = constant

$V = V_0 \quad , \quad \rho = a$

$V = 0 \quad\quad \rho = b \quad\quad b > a$

$$\boxed{V = V_0 \frac{\ln(b/\rho)}{\ln(b/a)}}$$

conducting sphere in a uniform field →



we are considering here a conducting uncharged sphere of radius R is placed in an uniform electric field $\vec{E} = E_0 \hat{z}$. this field pushes positive charge on the northern hemisphere, leaving a negative charge on the southern hemisphere.

the sphere is an equipotential.

In the entire xy plane is at $V=0$.

V does not go to zero at large z.

$$V = -\int E \cdot dS = \int E_0 \hat{z} \, dz$$

$$= -E_0 z + C$$

$$E_0 \hat{z} \cdot dz \hat{z}$$

$$\int E_0 \, dz$$

$$\boxed{V = -E_0 z + C}$$

① $V=0$ in equatorial plane. $C=0$

② $V = -E_0 z$ $\quad r = R$

$V = -E_0 r \cos\theta$ $\quad r \gg R$.

there are the boundary conditions for this problem.

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0$$

$$B_l = -A_l R^{2l+1}$$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l \cos\theta$$

For $r \gg R$

condition on ② requires that,

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta$$

$l=1$ $\quad P_1(\cos\theta) = \cos\theta$, $A_1 = -E_0$

all other A_l's zero.

so $\boxed{V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta}$

the first term $(-E_0 r \cos\theta)$ is due to the external field.

contribution due to the induced charges.

$$E_0 \frac{R^3}{r^2} \cos\theta$$

$$E = +\frac{\sigma}{\epsilon_0} \qquad \sigma = +\epsilon_0 E$$

$$\sigma(\theta) = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}$$

$$\epsilon_0 E_0 \left(1 + 2\frac{R^3}{r^3}\right)\cos\theta \Big|_{r=R}$$

$$\boxed{\sigma(\theta) \Rightarrow 3\epsilon_0 E_0 \cos\theta}$$

it is positive in the northern hemisphere $\left(0 \leq \theta \leq \frac{\pi}{2}\right)$ and negative in the southern hemisphere. $\left(\frac{\pi}{2} \leq \theta \leq \pi\right)$.