Transportation, Assignment, and Transshipment Problems

In this chapter, we discuss three special types of linear programming problems: transportation, assignment, and transshipment. Each of these can be solved by the simplex algorithm, but specialized algorithms for each type of problem are much more efficient.

7.1 Formulating Transportation Problems

We begin our discussion of transportation problems by formulating a linear programming model of the following situation.

EXAMPLE 1

Powerco has three electric power plants that supply the needs of four cities.[†] Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1—35 million; plant 2—50 million; plant 3—40 million (see Table 1). The peak power demands in these cities, which occur at the same time (2 P.M.), are as follows (in kwh): city 1—45 million; city 2—20 million; city 3—30 million; city 4—30 million. The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to minimize the cost of meeting each city's peak power demand.

Solution To formulate Powerco's problem as an LP, we begin by defining a variable for each decision that Powerco must make. Because Powerco must determine how much power is sent from each plant to each city, we define (for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$)

$$x_{ij} = \text{number of (million) kwh produced at plant } i \text{ and sent to city } j$$

In terms of these variables, the total cost of supplying the peak power demands to cities 1–4 may be written as

$$\begin{array}{lr}
8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} & \text{(Cost of shipping power from plant 1)} \\
+ \; 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} & \text{(Cost of shipping power from plant 2)} \\
+ \; 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} & \text{(Cost of shipping power from plant 3)}
\end{array}$$

Powerco faces two types of constraints. First, the total power supplied by each plant cannot exceed the plant's capacity. For example, the total amount of power sent from plant

---

[†] This example is based on Aarvik and Randolph (1975).

TABLE 1

Shipping Costs, Supply, and Demand for Powerco

	To				
From	**City 1**	**City 2**	**City 3**	**City 4**	**Supply (million kwh)**
Plant 1	\$8	\$6	\$10	\$9	35
Plant 2	\$9	\$12	\$13	\$7	50
Plant 3	\$14	\$9	\$16	\$5	40
Demand (million kwh)	45	20	30	30	

1 to the four cities cannot exceed 35 million kwh. Each variable with first subscript 1 represents a shipment of power from plant 1, so we may express this restriction by the LP constraint

$$x_{11} + x_{12} + x_{13} + x_{14} \le 35$$

In a similar fashion, we can find constraints that reflect plant 2's and plant 3's capacities. Because power is supplied by the power plants, each is a **supply point.** Analogously, a constraint that ensures that the total quantity shipped from a plant does not exceed plant capacity is a **supply constraint.** The LP formulation of Powerco's problem contains the following three supply constraints:

$$\begin{array}{ll}
x_{11} + x_{12} + x_{13} + x_{14} \le 35 & \text{(Plant 1 supply constraint)} \\
x_{21} + x_{22} + x_{23} + x_{24} \le 50 & \text{(Plant 2 supply constraint)} \\
x_{31} + x_{32} + x_{33} + x_{34} \le 40 & \text{(Plant 3 supply constraint)}
\end{array}$$

Second, we need constraints that ensure that each city will receive sufficient power to meet its peak demand. Each city demands power, so each is a **demand point.** For example, city 1 must receive at least 45 million kwh. Each variable with second subscript 1 represents a shipment of power to city 1, so we obtain the following constraint:

$$x_{11} + x_{21} + x_{31} \ge 45$$

Similarly, we obtain a constraint for each of cities 2, 3, and 4. A constraint that ensures that a location receives its demand is a **demand constraint.** Powerco must satisfy the following four demand constraints:

$$\begin{array}{ll}
x_{11} + x_{21} + x_{31} \ge 45 & \text{(City 1 demand constraint)} \\
x_{12} + x_{22} + x_{32} \ge 20 & \text{(City 2 demand constraint)} \\
x_{13} + x_{23} + x_{33} \ge 30 & \text{(City 3 demand constraint)} \\
x_{14} + x_{24} + x_{34} \ge 30 & \text{(City 4 demand constraint)}
\end{array}$$

Because all the x_{ij}'s must be nonnegative, we add the sign restrictions $x_{ij} \ge 0$ ($i = 1, 2, 3$; $j = 1, 2, 3, 4$).

Combining the objective function, supply constraints, demand constraints, and sign restrictions yields the following LP formulation of Powerco's problem:

$$\begin{array}{lll}
\min z = & 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} & \\
 & + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} & \\
\text{s.t.} & x_{11} + x_{12} + x_{13} + x_{14} \le 35 & \text{(Supply constraints)} \\
 & x_{21} + x_{22} + x_{23} + x_{24} \le 50 & \\
 & x_{31} + x_{32} + x_{33} + x_{34} \le 40 &
\end{array}$$

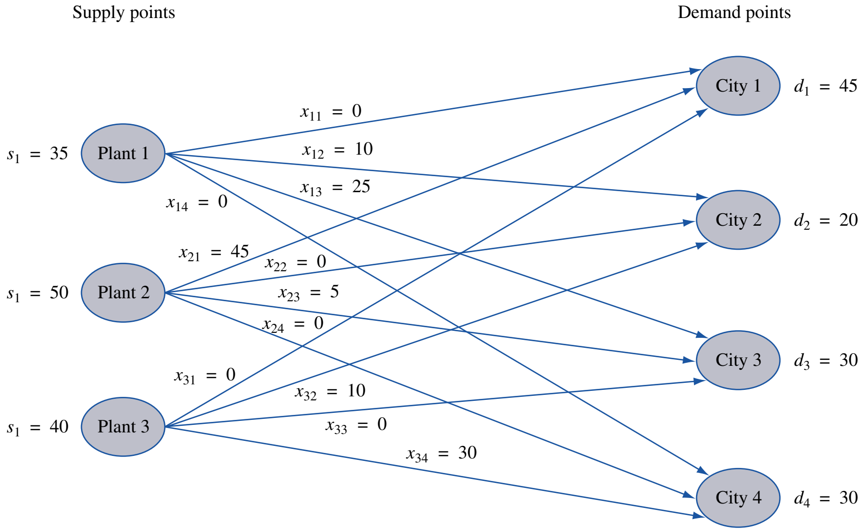


FIGURE 1
Graphical Representation of Powerco Problem and Its Optimal Solution

$$
\begin{aligned}
x_{11} + x_{21} + x_{31} &\geq 45 \qquad \text{(Demand constraints)}\\
x_{12} + x_{22} + x_{32} &\geq 20\\
x_{13} + x_{23} + x_{33} &\geq 30\\
x_{14} + x_{24} + x_{34} &\geq 30\\
x_{ij} &\geq 0 \quad (i = 1, 2, 3;\ j = 1, 2, 3, 4)
\end{aligned}
$$

In Section 7.3, we will find that the optimal solution to this LP is $z = 1020$, $x_{12} = 10$, $x_{13} = 25$, $x_{21} = 45$, $x_{23} = 5$, $x_{32} = 10$, $x_{34} = 30$. Figure 1 is a graphical representation of the Powerco problem and its optimal solution. The variable x_{ij} is represented by a line, or arc, joining the ith supply point (plant i) and the jth demand point (city j).

General Description of a Transportation Problem

In general, a transportation problem is specified by the following information:

1 A set of m *supply points* from which a good is shipped. Supply point i can supply at most s_i units. In the Powerco example, $m = 3$, $s_1 = 35$, $s_2 = 50$, and $s_3 = 40$.

2 A set of n *demand points* to which the good is shipped. Demand point j must receive at least d_j units of the shipped good. In the Powerco example, $n = 4$, $d_1 = 45$, $d_2 = 20$, $d_3 = 30$, and $d_4 = 30$.

3 Each unit produced at supply point i and shipped to demand point j incurs a *variable cost* of c_{ij}. In the Powerco example, $c_{12} = 6$.

Let

$$x_{ij} = \text{number of units shipped from supply point } i \text{ to demand point } j$$

then the general formulation of a transportation problem is

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{j=1}^{j=n} x_{ij} \le s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)} \\
& \sum_{i=1}^{i=m} x_{ij} \ge d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)} \\
& x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)
\end{aligned} \tag{1}$$

If a problem has the constraints given in (1) and is a *maximization* problem, then it is still a transportation problem (see Problem 7 at the end of this section). If

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j$$

then total supply equals total demand, and the problem is said to be a **balanced transportation problem.**

For the Powerco problem, total supply and total demand both equal 125, so this is a balanced transportation problem. In a balanced transportation problem, all the constraints must be binding. For example, in the Powerco problem, if any supply constraint were nonbinding, then the remaining available power would not be sufficient to meet the needs of all four cities. For a balanced transportation problem, (1) may be written as

$$\begin{aligned}
\min & \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{j=n} x_{ij} = s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)} \\
& \sum_{i=1}^{i=m} x_{ij} = d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)} \\
& x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)
\end{aligned} \tag{2}$$

Later in this chapter, we will see that it is relatively simple to find a basic feasible solution for a balanced transportation problem. Also, simplex pivots for these problems do not involve multiplication and reduce to additions and subtractions. For these reasons, it is desirable to formulate a transportation problem as a balanced transportation problem.

Balancing a Transportation Problem If Total Supply Exceeds Total Demand

If total supply exceeds total demand, we can balance a transportation problem by creating a **dummy demand point** that has a demand equal to the amount of excess supply. Because shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. Shipments to the dummy demand point indicate unused supply capacity. To understand the use of a dummy demand point, suppose that in the Powerco problem, the demand for city 1 were reduced to 40 million kwh. To balance the Powerco problem, we would add a dummy demand point (point 5) with a demand of $125 - 120 = 5$ million kwh. From each plant, the cost of shipping 1 million kwh to the dummy is 0. The optimal solution to this balanced transportation problem is $z = 975$, $x_{13} = 20$, $x_{12} = 15$, $x_{21} = 40$, $x_{23} = 10$, $x_{32} = 5$, $x_{34} = 30$, and $x_{35} = 5$. Because $x_{35} = 5$, 5 million kwh of plant 3 capacity will be unused (see Figure 2).

A transportation problem is specified by the supply, the demand, and the shipping costs, so the relevant data can be summarized in a **transportation tableau** (see Table 2). The square, or **cell,** in row i and column j of a transportation tableau corresponds to the

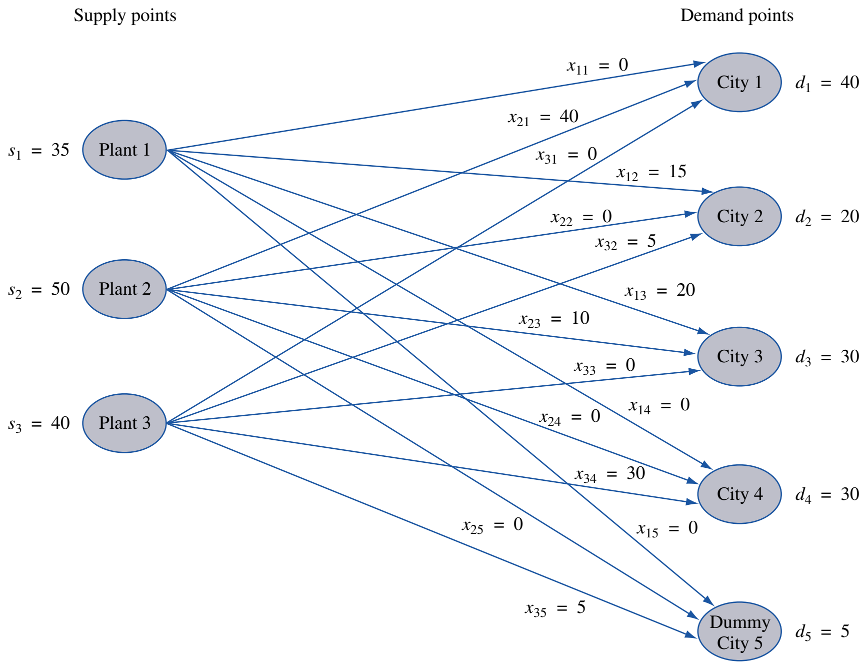


FIGURE 2
Graphical Representation of Unbalanced Powerco Problem and Its Optimal Solution (with Dummy Demand Point)

TABLE 2
A Transportation Tableau

					Supply
	c_{11}	c_{12}	. . .	c_{1n}	s_1
	c_{21}	c_{22}	. . .	c_{2n}	s_2
	⋮	⋮		⋮	⋮
	c_{m1}	c_{m2}	. . .	c_{mn}	s_m
Demand	d_1	d_2	. . .	d_n	

TABLE 3
Transportation Tableau for Powerco

	City 1	City 2	City 3	City 4	Supply
Plant 1	8	6 10	10 25	9	35
Plant 2	9 45	12	13 5	7	50
Plant 3	14	9 10	16	5 30	40
Demand	45	20	30	30	

variable x_{ij}. If x_{ij} is a basic variable, its value is placed in the lower left-hand corner of the *ij*th cell of the tableau. For example, the balanced Powerco problem and its optimal solution could be displayed as shown in Table 3. The tableau format implicitly expresses the supply and demand constraints through the fact that the sum of the variables in row i must equal s_i and the sum of the variables in column j must equal d_j.

Balancing a Transportation Problem If Total Supply Is Less Than Total Demand

If a transportation problem has a total supply that is strictly less than total demand, then the problem has no feasible solution. For example, if plant 1 had only 30 million kwh of capacity, then a total of only 120 million kwh would be available. This amount of power would be insufficient to meet the total demand of 125 million kwh, and the Powerco problem would no longer have a feasible solution.

When total supply is less than total demand, it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, a penalty is often associated with unmet demand. Example 2 illustrates how such a situation can yield a balanced transportation problem.

EXAMPLE 2 Handling Shortages

Two reservoirs are available to supply the water needs of three cities. Each reservoir can supply up to 50 million gallons of water per day. Each city would like to receive 40 million gallons per day. For each million gallons per day of unmet demand, there is a penalty. At city 1, the penalty is \$20; at city 2, the penalty is \$22; and at city 3, the penalty is \$23. The cost of transporting 1 million gallons of water from each reservoir to each city is shown in Table 4. Formulate a balanced transportation problem that can be used to minimize the sum of shortage and transport costs.

Solution In this problem,

$$\text{Daily supply} = 50 + 50 = 100 \text{ million gallons per day}$$
$$\text{Daily demand} = 40 + 40 + 40 = 120 \text{ million gallons per day}$$

To balance the problem, we add a dummy (or shortage) *supply point* having a supply of $120 - 100 = 20$ million gallons per day. The cost of shipping 1 million gallons from the dummy supply point to a city is just the shortage cost per million gallons for that city. Table 5 shows the balanced transportation problem and its optimal solution. Reservoir 1 should send 20 million gallons per day to city 1 and 30 million gallons per day to city 2, whereas reservoir 2 should send 10 million gallons per day to city 2 and 40 million gallons per day to city 3. Twenty million gallons per day of city 1's demand will be unsatisfied.

TABLE 4

Shipping Costs for Reservoir

	To		
From	**City 1**	**City 2**	**City 3**
Reservoir 1	\$7	\$8	\$10
Reservoir 2	\$9	\$7	\$8

TABLE 5
Transportation Tableau for Reservoir

	City 1	City 2	City 3	Supply
Reservoir 1	7 20	8 30	10	50
Reservoir 2	9	7 10	8 40	50
Dummy (shortage)	20 20	22	23	20
Demand	40	40	40	

Modeling Inventory Problems as Transportation Problems

Many inventory planning problems can be modeled as balanced transportation problems. To illustrate, we formulate a balanced transportation model of the Sailco problem of Section 3.10.

EXAMPLE 3 Setting Up an Inventory Problem as a Transportation Problem

Sailco Corporation must determine how many sailboats should be produced during each of the next four quarters (one quarter is three months). Demand is as follows: first quarter, 40 sailboats; second quarter, 60 sailboats; third quarter, 75 sailboats; fourth quarter, 25 sailboats. Sailco must meet demand on time. At the beginning of the first quarter, Sailco has an inventory of 10 sailboats. At the beginning of each quarter, Sailco must decide how many sailboats should be produced during the current quarter. For simplicity, we assume that sailboats manufactured during a quarter can be used to meet demand for the current quarter. During each quarter, Sailco can produce up to 40 sailboats at a cost of $400 per sailboat. By having employees work overtime during a quarter, Sailco can produce additional sailboats at a cost of $450 per sailboat. At the end of each quarter (after production has occurred and the current quarter's demand has been satisfied), a carrying or holding cost of $20 per sailboat is incurred. Formulate a balanced transportation problem to minimize the sum of production and inventory costs during the next four quarters.

Solution We define supply and demand points as follows:

Supply Points
Point 1 = initial inventory ($s_1 = 10$)
Point 2 = quarter 1 regular-time (RT) production ($s_2 = 40$)
Point 3 = quarter 1 overtime (OT) production ($s_3 = 150$)
Point 4 = quarter 2 RT production ($s_4 = 40$)
Point 5 = quarter 2 OT production ($s_5 = 150$)
Point 6 = quarter 3 RT production ($s_6 = 40$)
Point 7 = quarter 3 OT production ($s_7 = 150$)
Point 8 = quarter 4 RT production ($s_8 = 40$)
Point 9 = quarter 4 OT production ($s_9 = 150$)

There is a supply point corresponding to each source from which demand for sailboats can be met:

Point 1 = quarter 1 demand ($d_1 = 40$)
Point 2 = quarter 2 demand ($d_2 = 60$)
Demand Points Point 3 = quarter 3 demand ($d_3 = 75$)
Point 4 = quarter 4 demand ($d_4 = 25$)
Point 5 = dummy demand point ($d_5 = 770 - 200 = 570$)

A shipment from, say, quarter 1 RT to quarter 3 demand means producing 1 unit on regular time during quarter 1 that is used to meet 1 unit of quarter 3's demand. To determine, say, c_{13}, observe that producing 1 unit during quarter 1 RT and using that unit to meet quarter 3 demand incurs a cost equal to the cost of producing 1 unit on quarter 1 RT plus the cost of holding a unit in inventory for $3 - 1 = 2$ quarters. Thus, $c_{13} = 400 + 2(20) = 440$.

Because there is no limit on the overtime production during any quarter, it is not clear what value should be chosen for the supply at each overtime production point. Total demand = 200, so at most $200 - 10 = 190$ (-10 is for initial inventory) units will be produced during any quarter. Because 40 units must be produced on regular time before any units are produced on overtime, overtime production during any quarter will never exceed $190 - 40 = 150$ units. Any unused overtime capacity will be "shipped" to the dummy demand point. To ensure that no sailboats are used to meet demand during a quarter prior to their production, a cost of M (M is a large positive number) is assigned to any cell that corresponds to using production to meet demand for an earlier quarter.

TABLE 6
Transportation Tableau for Sailco

	1	2	3	4	Dummy	Supply
Initial	0 10	20	40	60	0	10
Qtr 1 RT	400 30	420 10	440	460	0	40
Qtr 1 OT	450	470	490	510	0 150	150
Qtr 2 RT	M	400 40	420	440	0	40
Qtr 2 OT	M	450 10	470	490	0 140	150
Qtr 3 RT	M	M	400 40	420	0	40
Qtr 3 OT	M	M	450 35	470	0 115	150
Qtr 4 RT	M	M	M	400 25	0 15	40
Qtr 4 OT	M	M	M	450	0 150	150
Demand	40	60	75	25	570	

Total supply = 770 and total demand = 200, so we must add a dummy demand point with a demand of 770 − 200 = 570 to balance the problem. The cost of shipping a unit from any supply point to the dummy demand point is 0.

Combining these observations yields the balanced transportation problem and its optimal solution shown in Table 6. Thus, Sailco should meet quarter 1 demand with 10 units of initial inventory and 30 units of quarter 1 RT production; quarter 2 demand with 10 units of quarter 1 RT, 40 units of quarter 2 RT, and 10 units of quarter 2 OT production; quarter 3 demand with 40 units of quarter 3 RT and 35 units of quarter 3 OT production; and finally, quarter 4 demand with 25 units of quarter 4 RT production.

In Problem 12 at the end of this section, we show how this formulation can be modified to incorporate other aspects of inventory problems (backlogged demand, perishable inventory, and so on).

Solving Transportation Problems on the Computer

To solve a transportation problem with LINDO, type in the objective function, supply constraints, and demand constraints. Other menu-driven programs are available that accept the shipping costs, supply values, and demand values. From these values, the program can generate the objective function and constraints.

Trans.lng

LINGO can be used to easily solve any transportation problem. The following LINGO model can be used to solve the Powerco example (file Trans.lng).

```
MODEL:
   1]SETS:
   2]PLANTS/P1,P2,P3/:CAP;
   3]CITIES/C1,C2,C3,C4/:DEM;
   4]LINKS(PLANTS,CITIES):COST,SHIP;
   5]ENDSETS
   6]MIN=@SUM(LINKS:COST*SHIP);
   7]@FOR(CITIES(J):
   8]@SUM(PLANTS(I):SHIP(I,J))>DEM(J));
   9]@FOR(PLANTS(I):
  10]@SUM(CITIES(J):SHIP(I,J))<CAP(I));
  11]DATA:
  12]CAP=35,50,40;
  13]DEM=45,20,30,30;
  14]COST=8,6,10,9,
  15]9,12,13,7,
  16]14,9,16,5;
  17]ENDDATA
END
```

Lines 1–5 define the **SETS** needed to generate the objective function and constraints. In line 2, we create the three power plants (the supply points) and specify that each has a capacity (given in the **DATA** section). In line 3, we create the four cities (the demand points) and specify that each has a demand (given in the **DATA** section). The **LINK** statement in line 4 creates a LINK(I,J) as I runs over all PLANTS and J runs over all CITIES. Thus, objects LINK(1,1), LINK (1,2), LINK(1,3), LINK(1,4), LINK(2,1), LINK (2,2), LINK(2,3), LINK(2,4), LINK(3,1), LINK (3,2), LINK(3,3), LINK(3,4) are created and stored in this order. Attributes with multiple subscripts are stored so that the rightmost subscripts advance most rapidly. Each LINK has two attributes: a per-unit shipping cost [(COST), given in the **DATA** section] and the amount shipped (SHIP), for which LINGO will solve.

Line 6 creates the objective function. We sum over all links the product of the unit shipping cost and the amount shipped. Using the **@FOR** and **@SUM** operators, lines 7–8 generate all demand constraints. They ensure that for each city, the sum of the amount shipped into the city will be at least as large as the city's demand. Note that the extra parenthesis after SHIP(I,J) in line 8 is to close the **@SUM** operator, and the extra parenthesis after DEM(J) is to close the **@FOR** operator. Using the **@FOR** and **@SUM** operators, lines 9–10 generate all supply constraints. They ensure that for each plant, the total shipped out of the plant will not exceed the plant's capacity.

Lines 11–17 contain the data needed for the problem. Line 12 defines each plant's capacity, and line 13 defines each city's demand. Lines 14–16 contain the unit shipping cost from each plant to each city. These costs correspond to the ordering of the links described previously. **ENDDATA** ends the data section, and **END** ends the program. Typing **GO** will solve the problem.

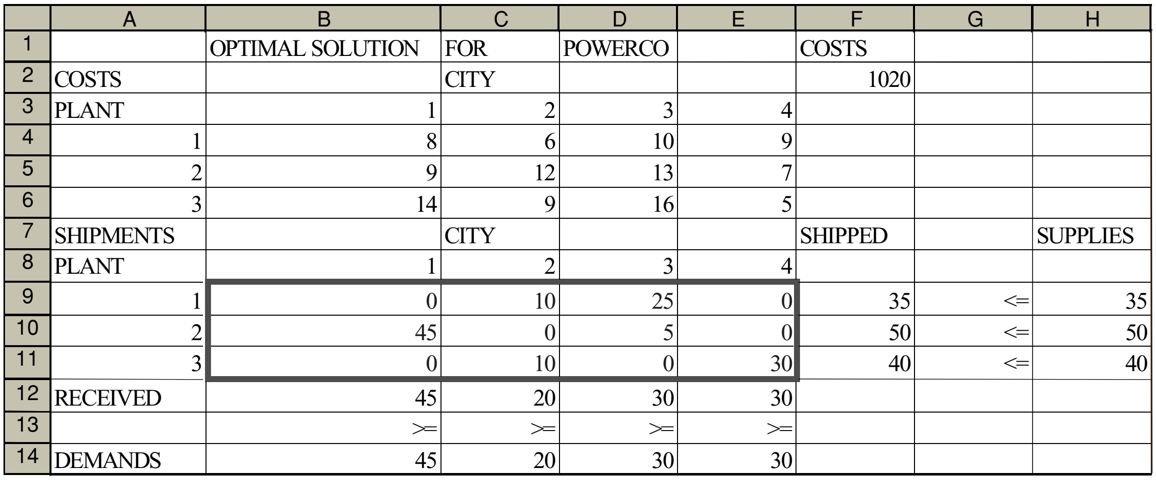
This program can be used to solve any transportation problem. If, for example, we wanted to solve a problem with 15 supply points and 10 demand points, we would change line 2 to create 15 supply points and line 3 to create 10 demand points. Moving to line 12, we would type in the 15 plant capacities. In line 13, we would type in the demands for the 10 demand points. Then in line 14, we would type in the 150 shipping costs. Observe that the part of the program (lines 6–10) that generates the objective function and constraints remains unchanged! Notice also that our LINGO formulation does not require that the transportation problem be balanced.

Obtaining LINGO Data from an Excel Spreadsheet

Powerco.xls

Often it is easier to obtain data for a LINGO model from a spreadsheet. For example, shipping costs for a transportation problem may be the end result of many computations. As an example, suppose we have created the capacities, demands, and shipping costs for the Powerco model in the file Powerco.xls (see Figure 3). We have created capacities in the cell range F9:F11 and named the range Cap. As you probably know, you can name a range of cells in Excel by selecting the range and clicking in the name box in the upper left-hand corner of your spreadsheet. Then type the range name and hit the Enter key. In a similar fashion, name the city demands (in cells B12:E12) with the name Demand and the unit shipping costs (in cells B4:E6) with the name Costs.

FIGURE 3



	A	B	C	D	E	F	G	H
1		OPTIMAL SOLUTION	FOR	POWERCO		COSTS		
2	COSTS		CITY			1020		
3	PLANT	1	2	3	4			
4	1	8	6	10	9			
5	2	9	12	13	7			
6	3	14	9	16	5			
7	SHIPMENTS		CITY			SHIPPED		SUPPLIES
8	PLANT	1	2	3	4			
9	1	0	10	25	0	35	<=	35
10	2	45	0	5	0	50	<=	50
11	3	0	10	0	30	40	<=	40
12	RECEIVED	45	20	30	30			
13		>=	>=	>=	>=			
14	DEMANDS	45	20	30	30			

Transpspread.lng

Using an **@OLE** statement, LINGO can read from a spreadsheet the values of data that are defined in the Sets portion of a program. The LINGO program (see file Transpspread.lng) needed to read our input data from the Powerco.xls file is shown below.

```
MODEL:
SETS:
PLANTS/P1,P2,P3/:CAP;
CITIES/C1,C2,C3,C4/:DEM;
LINKS(PLANTS,CITIES):COST,SHIP;
ENDSETS
MIN=@SUM(LINKS:COST*SHIP);
@FOR(CITIES(J):
@SUM(PLANTS(I):SHIP(I,J))>DEM(J));
@FOR(PLANTS(I);
@SUM(CITIES(J):SHIP(I,J))<CAP(I));
DATA:
CAP, DEM, COST=@OLE(‘C:\MPROG\POWERCO.XLS’,‘Cap’,‘Demand’,‘Costs’);
ENDDATA
  END
```

`The key statement is`

```
CAP, DEM, COST=@OLE(‘C:\MPROG\POWERCO.XLS’,‘Cap’,‘Demand’,‘Costs’);.
```

This statement reads the defined data sets CAP, DEM, and COSTS from the Powerco.xls spreadsheet. Note that the full path location of our Excel file (enclosed in single quotes) must be given first followed by the spreadsheet range names that contain the needed data. The range names are paired with the data sets in the order listed. Therefore, CAP values are found in range Cap and so on. The **@OLE** statement is very powerful, because a spreadsheet will usually greatly simplify the creation of data for a LINGO program.

Spreadsheet Solution of Transportation Problems

Powerco.xls

In the file Powerco.xls, we show how easy it is to use the Excel Solver to find the optimal solution to a transportation problem. After entering the plant capacities, city demands, and unit shipping costs as shown, we enter trial values of the units shipped from each plant to each city in the range B9:E11. Then we proceed as follows:

Step 1 Compute the total amount shipped out of each city by copying from F9 to F10:F11 the formula

=SUM(B9:E9)

Step 2 Compute the total received by each city by copying from B12 to C12:E12 the formula

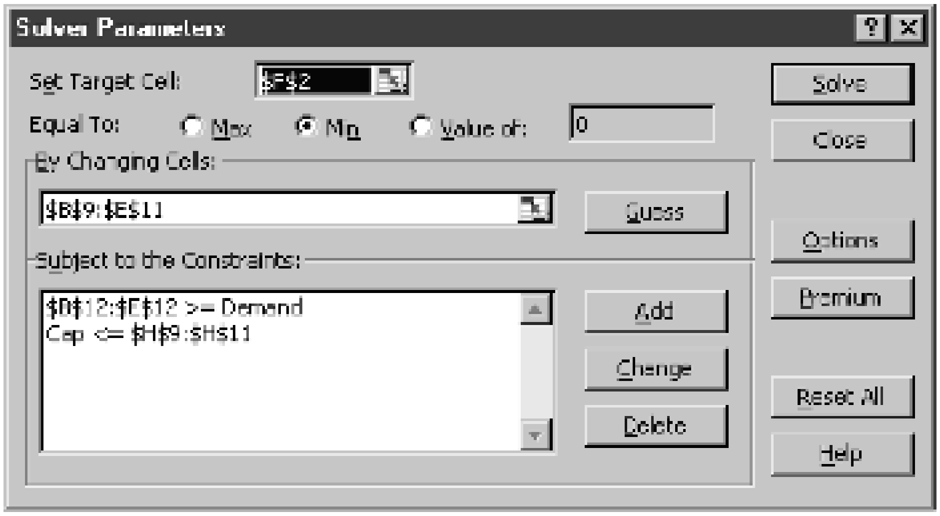
=SUM(B9:B11)

Step 3 Compute the total shipping cost in cell F2 with the formula

=SUMPRODUCT(B9:E11,Costs)

Note that the =SUMPRODUCT function works on rectangles as well as rows or columns of numbers. Also, we have named the range of unit shipping costs (B4:E6) as COSTS.

Step 4 We now fill in the Solver window shown in Figure 4. We minimize total shipping costs (F2) by changing units shipped from each plant to each city (B9:E11). We constrain amount received by each city (B12:E12) to be at least each city’s demand (range name Demand). We constrain the amount shipped out of each plant (F9:F11) to be at most each plant’s capacity (range name Cap). After checking the Assume Nonnegative option and Assume Linear Model option, we obtain the optimal solution shown in Figure 3. Note, of course, that the objective function of the optimal solution found by Excel equals the ob-



Set Target Cell: $F$2

Equal To: ○ Max ● Min ○ Value of: 0

By Changing Cells: $B$9:$E$11

Subject to the Constraints:

$B$12:$E$12 >= Demand
Cap <= $H$9:$H$11

FIGURE 4

jective function value found by LINGO and our hand solution. If the problem had multiple optimal solutions, then it is possible that the values of the shipments found by LINGO, Excel, and our hand solution might be different.

PROBLEMS

Group A

1 A company supplies goods to three customers, who each require 30 units. The company has two warehouses. Warehouse 1 has 40 units available, and warehouse 2 has 30 units available. The costs of shipping 1 unit from warehouse to customer are shown in Table 7. There is a penalty for each unmet customer unit of demand: With customer 1, a penalty cost of $90 is incurred; with customer 2, $80; and with customer 3, $110. Formulate a balanced transportation problem to minimize the sum of shortage and shipping costs.

2 Referring to Problem 1, suppose that extra units could be purchased and shipped to either warehouse for a total cost of $100 per unit and that all customer demand must be met. Formulate a balanced transportation problem to minimize the sum of purchasing and shipping costs.

3 A shoe company forecasts the following demands during the next six months: month 1—200; month 2—260; month 3—240; month 4—340; month 5—190; month 6—150. It costs $7 to produce a pair of shoes with regular-time labor (RT) and $11 with overtime labor (OT). During each month, regular production is limited to 200 pairs of shoes, and overtime production is limited to 100 pairs. It costs $1 per month to hold a pair of shoes in inventory. Formulate a balanced transportation problem to minimize the total cost of meeting the next six months of demand on time.

4 Steelco manufactures three types of steel at different plants. The time required to manufacture 1 ton of steel (regardless of type) and the costs at each plant are shown in Table 8. Each week, 100 tons of each type of steel (1, 2, and 3) must be produced. Each plant is open 40 hours per week.

a Formulate a balanced transportation problem to minimize the cost of meeting Steelco's weekly requirements.

b Suppose the time required to produce 1 ton of steel depends on the type of steel as well as on the plant at which it is produced (see Table 9, page 372). Could a transportation problem still be formulated?

5 A hospital needs to purchase 3 gallons of a perishable medicine for use during the current month and 4 gallons for use during the next month. Because the medicine is

TABLE 7

	To		
From	Customer 1	Customer 2	Customer 3
Warehouse 1	$15	$35	$25
Warehouse 2	$10	$50	$40

TABLE 8

	Cost ($)			
Plant	Steel 1	Steel 2	Steel 3	Time (minutes)
1	60	40	28	20
2	50	30	30	16
3	43	20	20	15

TABLE 9

	Time (minutes)		
Plant	**Steel 1**	**Steel 2**	**Steel 3**
1	15	12	15
2	15	15	20
3	10	10	15

perishable, it can only be used during the month of purchase. Two companies (Daisy and Laroach) sell the medicine. The medicine is in short supply. Thus, during the next two months, the hospital is limited to buying at most 5 gallons from each company. The companies charge the prices shown in Table 10. Formulate a balanced transportation model to minimize the cost of purchasing the needed medicine.

6 A bank has two sites at which checks are processed. Site 1 can process 10,000 checks per day, and site 2 can process 6,000 checks per day. The bank processes three types of checks: vendor, salary, and personal. The processing cost per check depends on the site (see Table 11). Each day, 5,000 checks of each type must be processed. Formulate a balanced transportation problem to minimize the daily cost of processing checks.

7† The U.S. government is auctioning off oil leases at two sites: 1 and 2. At each site, 100,000 acres of land are to be auctioned. Cliff Ewing, Blake Barnes, and Alexis Pickens are bidding for the oil. Government rules state that no bidder can receive more than 40% of the land being auctioned. Cliff has bid $1,000/acre for site 1 land and $2,000/acre for site 2 land. Blake has bid $900/acre for site 1 land and $2,200/acre for site 2 land. Alexis has bid $1,100/acre for site 1 land and $1,900/acre for site 2 land. Formulate a balanced transportation model to maximize the government's revenue.

TABLE 10

Company	**Current Month's Price per Gallon ($)**	**Next Month's Price per Gallon ($)**
Daisy	800	720
Laroach	710	750

TABLE 11

	Site (¢)	
Checks	**1**	**2**
Vendor	5	3
Salary	4	4
Personal	2	5

TABLE 12

	To ($)	
From ($)	**England**	**Japan**
Field 1	1	2
Field 2	2	1

TABLE 13

	Project ($)		
Auditor	**1**	**2**	**3**
1	120	150	190
2	140	130	120
3	160	140	150

8 The Ayatola Oil Company controls two oil fields. Field 1 can produce up to 40 million barrels of oil per day, and field 2 can produce up to 50 million barrels of oil per day. At field 1, it costs $3 to extract and refine a barrel of oil; at field 2, the cost is $2. Ayatola sells oil to two countries: England and Japan. The shipping cost per barrel is shown in Table 12. Each day, England is willing to buy up to 40 million barrels (at $6 per barrel), and Japan is willing to buy up to 30 million barrels (at $6.50 per barrel). Formulate a balanced transportation problem to maximize Ayatola's profits.

9 For the examples and problems of this section, discuss whether it is reasonable to assume that the proportionality assumption holds for the objective function.

10 Touche Young has three auditors. Each can work as many as 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours; project 2, 140 hours; and project 3, 160 hours. The amount per hour that can be billed for assigning each auditor to each project is given in Table 13. Formulate a balanced transportation problem to maximize total billings during the next month.

Group B

11‡ Paperco recycles newsprint, uncoated paper, and coated paper into recycled newsprint, recycled uncoated paper, and recycled coated paper. Recycled newsprint can be produced by processing newsprint or uncoated paper. Recycled coated paper can be produced by recycling any type of paper. Recycled uncoated paper can be produced by processing uncoated paper or coated paper. The process used to produce recycled newsprint removes 20% of the input's pulp, leaving 80% of the input's pulp for recycled paper. The process used to produce recycled coated paper removes 10% of the input's pulp. The process used to produce recycled uncoated paper removes 15% of the input's pulp. The purchasing costs, processing costs, and availability of each type of paper are shown in Table 14. To meet demand,

---

†This problem is based on Jackson (1980).

‡This problem is based on Glassey and Gupta (1974).

TABLE 14

	Purchase Cost per Ton of Pulp ($)	Processing Cost per Ton of Input ($)	Availability
Newsprint	10		500
Coated paper	9		300
Uncoated paper	8		200
NP used for RNP		3	
NP used for RCP		4	
UCP used for RNP		4	
UCP used for RUP		1	
UCP used for RCP		6	
CP used for RUP		5	
CP used for RCP		3	

Paperco must produce at least 250 tons of recycled newsprint pulp, at least 300 tons of recycled uncoated paper pulp, and at least 150 tons of recycled coated paper pulp. Formulate a balanced transportation problem that can be used to minimize the cost of meeting Paperco's demands.

12 Explain how each of the following would modify the formulation of the Sailco problem as a balanced transportation problem:

a Suppose demand could be backlogged at a cost of $30/sailboat/month. (*Hint:* Now it is permissible to ship from, say, month 2 production to month 1 demand.)

b If demand for a sailboat is not met on time, the sale is lost and an opportunity cost of $450 is incurred.

c Sailboats can be held in inventory for a maximum of two months.

d At a cost of $440/sailboat, Sailco can purchase up to 10 sailboats/month from a subcontractor.

7.2 Finding Basic Feasible Solutions for Transportation Problems

Consider a balanced transportation problem with m supply points and n demand points. From (2), we see that such a problem contains $m + n$ equality constraints. From our experience with the Big M method and the two-phase simplex method, we know it is difficult to find a bfs if all of an LP's constraints are equalities. Fortunately, the special structure of a balanced transportation problem makes it easy for us to find a bfs.

Before describing three methods commonly used to find a bfs to a balanced transportation problem, we need to make the following important observation. *If a set of values for the x_{ij}'s satisfies all but one of the constraints of a balanced transportation problem, then the values for the x_{ij}'s will automatically satisfy the other constraint.* For example, in the Powerco problem, suppose a set of values for the x_{ij}'s is known to satisfy all the constraints with the exception of the first supply constraint. Then this set of x_{ij}'s must supply $d_1 + d_2 + d_3 + d_4 = 125$ million kwh to cities 1–4 and supply $s_2 + s_3 = 125 - s_1 = 90$ million kwh from plants 2 and 3. Thus, plant 1 must supply $125 - (125 - s_1) = 35$ million kwh, so the x_{ij}'s must also satisfy the first supply constraint.

The preceding discussion shows that when we solve a balanced transportation problem, we may omit from consideration any one of the problem's constraints and solve an LP having $m + n - 1$ constraints. We (arbitrarily) assume that the first supply constraint is omitted from consideration.

In trying to find a bfs to the remaining $m + n - 1$ constraints, you might think that any collection of $m + n - 1$ variables would yield a basic solution. Unfortunately, this is not the case. For example, consider (3), a balanced transportation problem. (We omit the costs because they are not needed to find a bfs.)

			4
			5
3	2	4	

(3)

In matrix form, the constraints for this balanced transportation problem may be written as

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 2 \\ 4 \end{bmatrix} \tag{3'}$$

After dropping the first supply constraint, we obtain the following linear system:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \end{bmatrix} \tag{3''}$$

A basic solution to (3″) must have four basic variables. Suppose we try BV $= \{x_{11}, x_{12}, x_{21}, x_{22}\}$. Then

$$B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For $\{x_{11}, x_{12}, x_{21}, x_{22}\}$ to yield a basic solution, it must be possible to use EROs to transform B to I_4. Because rank $B = 3$ and EROs do not change the rank of a matrix, there is no way that EROs can be used to transform B into I_4. Thus, BV $= \{x_{11}, x_{12}, x_{21}, x_{22}\}$ cannot yield a basic solution to (3″). Fortunately, the simple concept of a loop may be used to determine whether an arbitrary set of $m + n - 1$ variables yields a basic solution to a balanced transportation problem.

DEFINITION ■ An ordered sequence of at least four different cells is called a **loop** if

1 Any two consecutive cells lie in either the same row or same column

2 No three consecutive cells lie in the same row or column

3 The last cell in the sequence has a row or column in common with the first cell in the sequence ■

In the definition of a loop, the first cell is considered to follow the last cell, so the loop may be thought of as a closed path. Here are some examples of the preceding definition: Figure 5 represents the loop (2, 1)–(2, 4)–(4, 4)–(4, 1). Figure 6 represents the loop (1, 1)–(1, 2)–(2, 2)–(2, 3)–(4, 3)–(4, 5)–(3, 5)–(3, 1). In Figure 7, the path (1, 1)–(1, 2)–(2, 3)–(2, 1) does not represent a loop, because (1, 2) and (2, 3) do not lie in the same row or column. In Figure 8, the path (1, 2)–(1, 3)–(1, 4)–(2, 4)–(2, 2) does not represent a loop, because (1, 2), (1, 3), and (1, 4) all lie in the same row.

Theorem 1 (which we state without proof) shows why the concept of a loop is important.

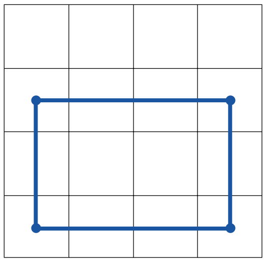


FIGURE 5

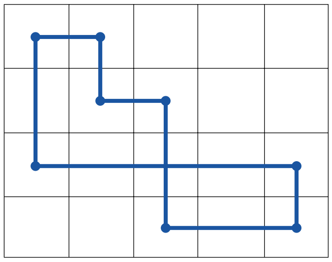


FIGURE 6

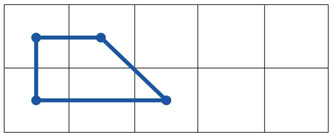


FIGURE 7

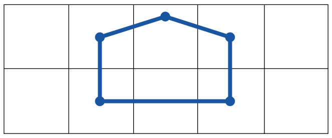


FIGURE 8

THEOREM 1

In a balanced transportation problem with m supply points and n demand points, the cells corresponding to a set of $m + n - 1$ variables contain no loop if and only if the $m + n - 1$ variables yield a basic solution.

Theorem 1 follows from the fact that a set of $m + n - 1$ cells contains no loop if and only if the $m + n - 1$ columns corresponding to these cells are linearly independent. Because (1, 1)–(1, 2)–(2, 2)–(2, 1) is a loop, Theorem 1 tells us that $\{x_{11}, x_{12}, x_{22}, x_{21}\}$ cannot yield a basic solution for (3″). On the other hand, no loop can be formed with the cells (1, 1)–(1, 2)–(1, 3)–(2, 1), so $\{x_{11}, x_{12}, x_{13}, x_{21}\}$ will yield a basic solution to (3″).

We are now ready to discuss three methods that can be used to find a basic feasible solution for a balanced transportation problem:

1 northwest corner method

2 minimum-cost method

3 Vogel's method

Northwest Corner Method for Finding a Basic Feasible Solution

To find a bfs by the northwest corner method, we begin in the upper left (or northwest) corner of the transportation tableau and set x_{11} as large as possible. Clearly, x_{11} can be no larger than the smaller of s_1 and d_1. If $x_{11} = s_1$, cross out the first row of the transportation tableau; this indicates that no more basic variables will come from row 1. Also change d_1 to $d_1 - s_1$. If $x_{11} = d_1$, cross out the first column of the transportation tableau; this indicates that no more basic variables will come from column 1. Also change s_1 to $s_1 - d_1$. If $x_{11} = s_1 = d_1$, cross out either row 1 or column 1 (but not both). If you cross out row 1, change d_1 to 0; if you cross out column 1, change s_1 to 0.

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed-out row or column. Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row and column. A basic feasible solution has now been obtained.

We illustrate the use of the northwest corner method by finding a bfs for the balanced transportation problem in Table 15. (We do not list the costs because they are not needed to apply the algorithm.) We indicate the crossing out of a row or column by placing an × by the row's supply or column's demand.

To begin, we set $x_{11} = \min\{5, 2\} = 2$. Then we cross out column 1 and change s_1 to $5 - 2 = 3$. This yields Table 16. The most northwest remaining variable is x_{12}. We set $x_{12} = \min\{3, 4\} = 3$. Then we cross out row 1 and change d_2 to $4 - 3 = 1$. This yields Table 17. The most northwest available variable is now x_{22}. We set $x_{22} = \min\{1, 1\} = 1$. Because both the supply and demand corresponding to the cell are equal, we may cross out either row 2 or column 2 (but not both). For no particular reason, we choose to cross out row 2. Then d_2 must be changed to $1 - 1 = 0$. The resulting tableau is Table 18. At the next step, this will lead to a *degenerate* bfs.

TABLE 15

				5
				1
				3
2	4	2	1	

TABLE 16

2				3
				1
				3
×	4	2	1	

TABLE 17

2	3			×
				1
				3
×	1	2	1	

TABLE 18

2	3			×
	1			×
				3
×	0	2	1	

The most northwest available cell is now x_{32}, so we set $x_{32} = \min\{3, 0\} = 0$. Then we cross out column 2 and change s_3 to $3 - 0 = 3$. The resulting tableau is Table 19. We now set $x_{33} = \min\{3, 2\} = 2$. Then we cross out column 3 and reduce s_3 to $3 - 2 = 1$. The resulting tableau is Table 20. The only available cell is x_{34}. We set $x_{34} = \min\{1, 1\} = 1$. Then we cross out row 3 and column 4. No cells are available, so we are finished. We have obtained the bfs $x_{11} = 2$, $x_{12} = 3$, $x_{22} = 1$, $x_{32} = 0$, $x_{33} = 2$, $x_{34} = 1$.

Why does the northwest corner method yield a bfs? The method ensures that no basic variable will be assigned a negative value (because no right-hand side ever becomes nega-

TABLE 19

2	3			×
	1			×
	0			3
×	×	2	1	

TABLE 20

2	3			×
	1			×
	0	2		1
×	×	×	1	

tive) and also that each supply and demand constraint is satisfied (because every row and column is eventually crossed out). Thus, the northwest corner method yields a feasible solution.

To complete the northwest corner method, $m + n$ rows and columns must be crossed out. The last variable assigned a value results in a row and column being crossed out, so the northwest corner method will assign values to $m + n - 1$ variables. The variables chosen by the northwest corner method cannot form a loop, so Theorem 1 implies that the northwest corner method must yield a bfs.

Minimum-Cost Method for Finding a Basic Feasible Solution

The northwest corner method does not utilize shipping costs, so it can yield an initial bfs that has a very high shipping cost. Then determining an optimal solution may require several pivots. The minimum-cost method uses the shipping costs in an effort to produce a bfs that has a lower total cost. Hopefully, fewer pivots will then be required to find the problem's optimal solution.

To begin the minimum-cost method, find the variable with the smallest shipping cost (call it x_{ij}). Then assign x_{ij} its largest possible value, $\min\{s_i, d_j\}$. As in the northwest corner method, cross out row i or column j and reduce the supply or demand of the noncrossed-out row or column by the value of x_{ij}. Then choose from the cells that do not lie in a crossed-out row or column the cell with the minimum shipping cost and repeat the procedure. Continue until there is only one cell that can be chosen. In this case, cross out both the cell's row and column. Remember that (with the exception of the last variable) if a variable satisfies both a supply and demand constraint, only cross out a row or column, not both.

To illustrate the minimum cost method, we find a bfs for the balanced transportation problem in Table 21. The variable with the minimum shipping cost is x_{22}. We set $x_{22} = \min\{10, 8\} = 8$. Then we cross out column 2 and reduce s_2 to $10 - 8 = 2$ (Table 22). We could now choose either x_{11} or x_{21} (both having shipping costs of 2). We arbitrarily choose x_{21} and set $x_{21} = \min\{2, 12\} = 2$. Then we cross out row 2 and change d_1 to $12 - 2 = 10$ (Table 23). Now we set $x_{11} = \min\{5, 10\} = 5$, cross out row 1, and change d_1 to $10 - 5 = 5$ (Table 24). The minimum cost that does not lie in a crossed-out row or column is x_{31}. We set $x_{31} = \min\{15, 5\} = 5$, cross out column 1, and reduce s_3 to $15 - 5 = 10$ (Table 25). Now we set $x_{33} = \min\{10, 4\} = 4$, cross out column 3, and reduce s_3 to $10 - 4 = 6$ (Table 26). The only cell that we can choose is x_{34}. We set $x_{34} = \min\{6, 6\}$ and cross out both row 3 and column 4. We have now obtained the bfs: $x_{11} = 5$, $x_{21} = 2$, $x_{22} = 8$, $x_{31} = 5$, $x_{33} = 4$, and $x_{34} = 6$.

Because the minimum-cost method chooses variables with small shipping costs to be basic variables, you might think that this method would always yield a bfs with a relatively low total shipping cost. The following problem shows how the minimum-cost method can be fooled into choosing a relatively high-cost bfs.

TABLE 21

2	3	5	6	5
2	1	3	5	10
3	8	4	6	15
12	8	4	6	

TABLE 22

$\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	5
$\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	2
$\boxed{3}$	$\boxed{8}$	$\boxed{4}$	$\boxed{6}$	15
12	×	4	6	

TABLE 23

$\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	5
2 $\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	×
$\boxed{3}$	$\boxed{8}$	$\boxed{4}$	$\boxed{6}$	15
10	×	4	6	

TABLE 24

5 $\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	×
2 $\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	×
$\boxed{3}$	$\boxed{8}$	$\boxed{4}$	$\boxed{6}$	15
5	×	4	6	

TABLE 25

5 $\boxed{2}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	×
2 $\boxed{2}$	8 $\boxed{1}$	$\boxed{3}$	$\boxed{5}$	×
5 $\boxed{3}$	$\boxed{8}$	$\boxed{4}$	$\boxed{6}$	10
×	×	4	6	

TABLE 26

2 5	3	5	6	×
2 2	1 8	3	5	×
3 5	8	4 4	6	6
×	×	×	6	

TABLE 27

6	7	8	10
15	80	78	15
15	5	5	

If we apply the minimum-cost method to Table 27, we set $x_{11} = 10$ and cross out row 1. This forces us to make x_{22} and x_{23} basic variables, thereby incurring their high shipping costs. Thus, the minimum-cost method will yield a costly bfs. Vogel's method for finding a bfs usually avoids extremely high shipping costs.

Vogel's Method for Finding a Basic Feasible Solution

Begin by computing for each row (and column) a "penalty" equal to the difference between the two smallest costs in the row (column). Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest shipping cost. As described in the northwest corner and minimum-cost methods, make this variable as large as possible, cross out a row or column, and change the supply or demand associated with the basic variable. Now recompute new penalties (using only cells that do not lie in a crossed-out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable, and cross out the variable's row and column. A bfs has now been obtained.

We illustrate Vogel's method by finding a bfs to Table 28. Column 2 has the largest penalty, so we set $x_{12} = \min\{10, 5\} = 5$. Then we cross out column 2 and reduce s_1 to $10 - 5 = 5$. After recomputing the new penalties (observe that after a column is crossed out, the column penalties will remain unchanged), we obtain Table 29. The largest penalty now occurs in column 3, so we set $x_{13} = \min\{5, 5\}$. We may cross out either row 1 or column 3. We arbitrarily choose to cross out column 3, and we reduce s_1 to $5 - 5 = 0$. Because each row has only one cell that is not crossed out, there are no row penalties. The resulting tableau is Table 30. Column 1 has the only (and, of course, the largest) penalty. We set $x_{11} = \min\{0, 15\} = 0$, cross out row 1, and change d_1 to $15 - 0 = 15$. The result is Table 31. No penalties can be computed, and the only cell that is not in a crossed-out row or column is x_{21}. Therefore, we set $x_{21} = 15$ and cross out both column 1 and row 2. Our application of Vogel's method is complete, and we have obtained the bfs: $x_{11} = 0$, $x_{12} = 5$, $x_{13} = 5$, and $x_{21} = 15$ (see Table 32).

TABLE 28

				Supply	Row Penalty
	6	7	8	10	7 – 6 = 1
	15	80	78	15	78 – 15 = 63
Demand	15	5	5		
Column Penalty	15 – 6 = 9	80 – 7 = 73	78 – 8 = 70		

TABLE 29

				Supply	Row Penalty
	6	7 5	8	5	8 – 6 = 2
	15	80	78	15	78 – 15 = 63
Demand	15	×	5		
Column Penalty	9	—	70		

TABLE 30

				Supply	Row Penalty
	6	7 5	8 5	0	—
	15	80	78	15	—
Demand	15	×	×		
Column Penalty	9	—	—		

TABLE 31

				Supply	Row Penalty
	6 0	7 5	8 5	×	—
	15	80	78	15	—
Demand	15	×	×		
Column Penalty	—	—	—		

TABLE 32

6 0	7 5	8 5	10
15 15	80	78	15
15	5	5	

Observe that Vogel's method avoids the costly shipments associated with x_{22} and x_{23}. This is because the high shipping costs resulted in large penalties that caused Vogel's method to choose other variables to satisfy the second and third demand constraints.

Of the three methods we have discussed for finding a bfs, the northwest corner method requires the least effort, and Vogel's method requires the most effort. Extensive research [Glover et al. (1974)] has shown, however, that when Vogel's method is used to find an initial bfs, it usually takes substantially fewer pivots than if the other two methods had been used. For this reason, the northwest corner and minimum-cost methods are rarely used to find a basic feasible solution to a large transportation problem.

PROBLEMS

Group A

1 Use the northwest corner method to find a bfs for Problems 1, 2, and 3 of Section 7.1.

2 Use the minimum-cost method to find a bfs for Problems 4, 7, and 8 of Section 7.1. (*Hint:* For a maximization problem, call the minimum-cost method the maximum-profit method or the maximum-revenue method.)

3 Use Vogel's method to find a bfs for Problems 5 and 6 of Section 7.1.

4 How should Vogel's method be modified to solve a maximization problem?

7.3 The Transportation Simplex Method

In this section, we show how the simplex algorithm simplifies when a transportation problem is solved. We begin by discussing the pivoting procedure for a transportation problem.

Recall that when the pivot row was used to eliminate the entering basic variable from other constraints and row 0, many multiplications were usually required. In solving a transportation problem, however, *pivots require only additions and subtractions.*

How to Pivot in a Transportation Problem

By using the following procedure, the pivots for a transportation problem may be performed within the confines of the transportation tableau:

Step 1 Determine (by a criterion to be developed shortly) the variable that should enter the basis.

Step 2 Find the loop (it can be shown that there is only one loop) involving the entering variable and some of the basic variables.

Step 3 Counting *only cells in the loop,* label those found in step 2 that are an even number (0, 2, 4, and so on) of cells away from the entering variable as *even* cells. Also label those that are an odd number of cells away from the entering variable as *odd* cells.

Step 4 Find the odd cell whose variable assumes the smallest value. Call this value θ. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by θ and increase the value of each even cell by θ. The values of variables not in the loop remain unchanged. The pivot is now complete. If $\theta = 0$, then the entering variable will equal 0, and an odd variable that has a current value of 0 will leave the basis. In this case, a degenerate bfs existed before and will result after the pivot. If more than one odd cell in the loop equals θ, you may arbitrarily choose one of these odd cells to leave the basis; again, a degenerate bfs will result.

We illustrate the pivoting procedure on the Powerco example. When the northwest corner method is applied to the Powerco example, the bfs in Table 33 is found. For this bfs, the basic variables are $x_{11} = 35$, $x_{21} = 10$, $x_{22} = 20$, $x_{23} = 20$, $x_{33} = 10$, and $x_{34} = 30$.

Suppose we want to find the bfs that would result if x_{14} were entered into the basis. The loop involving x_{14} and some of the basic variables is

$$\begin{array}{cccccc} \text{E} & \text{O} & \text{E} & \text{O} & \text{E} & \text{O} \\ (1, 4)– & (3, 4)– & (3, 3)– & (2, 3)– & (2, 1)– & (1, 1) \end{array}$$

In this loop, (1, 4), (3, 3), and (2, 1) are the even cells, and (1, 1), (3, 4), and (2, 3) are the odd cells. The odd cell with the smallest value is $x_{23} = 20$. Thus, after the pivot, x_{23} will have left the basis. We now add 20 to each of the even cells and subtract 20 from each of the odd cells. The bfs in Table 34 results. Because each row and column has as many $+20$s as -20s, the new solution will satisfy each supply and demand constraint. By choosing the smallest odd variable (x_{23}) to leave the basis, we have ensured that all variables will remain nonnegative. Thus, the new solution is feasible. There is no loop involving the cells (1, 1), (1, 4), (2, 1), (2, 2), (3, 3), and (3, 4), so the new solution is a bfs. After the pivot, the new bfs is $x_{11} = 15$, $x_{14} = 20$, $x_{21} = 30$, $x_{22} = 20$, $x_{33} = 30$, and $x_{34} = 10$, and all other variables equal 0.

TABLE 33

Northwest Corner Basic Feasible Solution for Powerco

35				35
10	20	20		50
		10	30	40
45	20	30	30	

TABLE 34

New Basic Feasible Solution After x_{14} Is Pivoted into Basis

35 – 20			0 + 20	35
10 + 20	20	20 – 20 (nonbasic)		50
		10 + 20	30 – 20	40
45	20	30	30	

The preceding illustration of the pivoting procedure makes it clear that each pivot in a transportation problem involves only additions and subtractions. Using this fact, we can show that *if all the supplies and demands for a transportation problem are integers, then the transportation problem will have an optimal solution in which all the variables are integers.* Begin by observing that, by the northwest corner method, we can find a bfs in which each variable is an integer. Each pivot involves only additions and subtractions, so each bfs obtained by performing the simplex algorithm (including the optimal solution) will assign all variables integer values. The fact that a transportation problem with integer supplies and demands has an optimal integer solution is useful, because it ensures that we need not worry about whether the Divisibility Assumption is justified.

Pricing Out Nonbasic Variables (Based on Chapter 6)

To complete our discussion of the transportation simplex, we now show how to compute row 0 for any bfs. From Section 6.2, we know that for a bfs in which the set of basic variables is BV, the coefficient of the variable x_{ij} (call it $\bar{c}_{ij}$) in the tableau's row 0 is given by

$$\bar{c}_{ij} = \mathbf{c}_{\mathrm{BV}}B^{-1}\mathbf{a}_{ij} - c_{ij}$$

where c_{ij} is the objective function coefficient for x_{ij} and $\mathbf{a}_{ij}$ is the column for x_{ij} in the original LP (we are assuming that the first supply constraint has been dropped).

Because we are solving a minimization problem, the current bfs will be optimal if all the $\bar{c}_{ij}$'s are nonpositive; otherwise, we enter into the basis the variable with the most positive $\bar{c}_{ij}$.

After determining $\mathbf{c}_{\mathrm{BV}}B^{-1}$, we can easily determine $\bar{c}_{ij}$. Because the first constraint has been dropped, $\mathbf{c}_{\mathrm{BV}}B^{-1}$ will have $m + n - 1$ elements. We write

$$\mathbf{c}_{\mathrm{BV}}B^{-1} = [u_2 \quad u_3 \quad \cdots \quad u_m \quad v_1 \quad v_2 \quad \cdots \quad v_n]$$

where $u_2, u_3, \ldots, u_m$ are the elements of $\mathbf{c}_{\mathrm{BV}}B^{-1}$ corresponding to the $m - 1$ supply constraints, and $v_1, v_2, \ldots, v_n$ are the elements of $\mathbf{c}_{\mathrm{BV}}B^{-1}$ corresponding to the n demand constraints.

To determine $\mathbf{c}_{\mathrm{BV}}B^{-1}$, we use the fact that in any tableau, each basic variable x_{ij} must have $\bar{c}_{ij} = 0$. Thus, for each of the $m + n - 1$ variables in BV,

$$\mathbf{c}_{\mathrm{BV}}B^{-1}\mathbf{a}_{ij} - c_{ij} = 0 \tag{4}$$

For a transportation problem, the equations in (4) are very easy to solve. To illustrate the solution of (4), we find $\mathbf{c}_{\mathrm{BV}}B^{-1}$ for (5), by applying the northwest corner method bfs to the Powerco problem.

8 35	6	10	9	35
9 10	12 20	13 20	7	50
14	9	16 10	5 30	40
45	20	30	30	

(5)

For this bfs, BV $= \{x_{11}, x_{21}, x_{22}, x_{23}, x_{33}, x_{34}\}$. Applying (4) we obtain

$$\bar{c}_{11} = [u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3 \quad v_4]\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} - 8 = v_1 - 8 = 0$$

$$\bar{c}_{21} = [u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3 \quad v_4]\begin{bmatrix}1\\0\\1\\0\\0\\0\end{bmatrix} - 9 = u_2 + v_1 - 9 = 0$$

$$\bar{c}_{22} = [u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3 \quad v_4]\begin{bmatrix}1\\0\\0\\1\\0\\0\end{bmatrix} - 12 = u_2 + v_2 - 12 = 0$$

$$\bar{c}_{23} = [u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3 \quad v_4]\begin{bmatrix}1\\0\\0\\0\\1\\0\end{bmatrix} - 13 = u_2 + v_3 - 13 = 0$$

$$\bar{c}_{33} = [u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3 \quad v_4]\begin{bmatrix}0\\1\\0\\0\\1\\0\end{bmatrix} - 16 = u_3 + v_3 - 16 = 0$$

$$\bar{c}_{34} = [u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3 \quad v_4]\begin{bmatrix}0\\1\\0\\0\\0\\1\end{bmatrix} - 5 = u_3 + v_4 - 5 = 0$$

For each basic variable x_{ij} (except those having $i = 1$), we see that (4) reduces to $u_i + v_j = c_{ij}$. If we define $u_1 = 0$, we see that (4) reduces to $u_i + v_j = c_{ij}$ for all basic variables. Thus, to solve for $\mathbf{c}_{\mathrm{BV}}B^{-1}$, we must solve the following system of $m + n$ equations: $u_1 = 0$, $u_i + v_j = c_{ij}$ for all basic variables.

For (5), we find $\mathbf{c}_{\mathrm{BV}}B^{-1}$ by solving

$$u_1 = 0 \tag{6}$$
$$u_1 + v_1 = 8 \tag{7}$$
$$u_2 + v_1 = 9 \tag{8}$$
$$u_2 + v_2 = 12 \tag{9}$$
$$u_2 + v_3 = 13 \tag{10}$$
$$u_3 + v_3 = 16 \tag{11}$$
$$u_3 + v_4 = 5 \tag{12}$$

From (7), $v_1 = 8$. From (8), $u_2 = 1$. Then (9) yields $v_2 = 11$, and (10) yields $v_3 = 12$. From (11), $u_3 = 4$. Finally, (12) yields $v_4 = 1$. For each nonbasic variable, we now compute $\bar{c}_{ij} = u_i + v_j - c_{ij}$. We obtain

$$\begin{aligned} \bar{c}_{12} &= 0 + 11 - 6 = 5 & \bar{c}_{13} &= 0 + 12 - 10 = 2 \\ \bar{c}_{14} &= 0 + 1 - 9 = -8 & \bar{c}_{24} &= 1 + 1 - 7 = -5 \\ \bar{c}_{31} &= 4 + 8 - 14 = -2 & \bar{c}_{32} &= 4 + 11 - 9 = 6 \end{aligned}$$

Because $\bar{c}_{32}$ is the most positive $\bar{c}_{ij}$, we would next enter x_{32} into the basis. Each unit of x_{32} that is entered into the basis will decrease Powerco's cost by \$6.

How to Determine the Entering Nonbasic Variable (Based on Chapter 5)

For readers who have not covered Chapter 6, we now discuss how to determine whether a bfs is optimal, and, if it is not, how to determine which nonbasic variable should enter the basis. Let $-u_i$ $(i = 1, 2, \ldots, m)$ be the shadow price of the ith supply constraint, and let $-v_j$ $(j = 1, 2, \ldots, n)$ be the shadow price of the jth demand constraint. We assume that the first supply constraint has been dropped, so we may set $-u_1 = 0$. From the definition of shadow price, if we were to increase the right-hand side of the ith supply and jth demand constraint by 1, the optimal z-value would decrease by $-u_i - v_j$. Equivalently, if we were to decrease the right-hand side of the ith supply and jth demand constraint by 1, the optimal z-value would increase by $-u_i - v_j$. Now suppose x_{ij} is a nonbasic variable. Should we enter x_{ij} into the basis? Observe that if we increase x_{ij} by 1, costs directly increase by c_{ij}. Also, increasing x_{ij} by 1 means that one less unit will be shipped from supply point i and one less unit will be shipped to demand point j. This is equivalent to reducing the right-hand sides of the ith supply constraint and jth demand constraint by 1. This will increase z by $-u_i - v_j$. Thus, increasing x_{ij} by 1 will increase z by a total of $c_{ij} - u_i - v_j$. So if $c_{ij} - u_i - v_j \geq 0$ (or $u_i + v_j - c_{ij} \leq 0$) for all nonbasic variables, the current bfs will be optimal. If, however, a nonbasic variable x_{ij} has $c_{ij} - u_i - v_j < 0$ (or $u_i + v_j - c_{ij} > 0$), then z can be decreased by $u_i + v_j - c_{ij}$ per unit of x_{ij} by entering x_{ij} into the basis. Thus, we may conclude that if $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, the nonbasic variable with the most positive value of $u_i + v_j - c_{ij}$ should enter the basis. How do we find the u_i's and v_j's? The coefficient of a nonbasic variable x_{ij} in row 0 of any tableau is the amount by which a unit increase in x_{ij} will decrease z, so we can conclude that the coefficient of any nonbasic variable (and, it turns out, any basic variable) in row 0 is $u_i + v_j - c_{ij}$. So we may solve for the u_i's and v_j's by solving the following system of equations: $u_1 = 0$ and $u_i + v_j - c_{ij} = 0$ for all basic variables.

To illustrate the previous discussion, consider the bfs for the Powerco problem shown in (5).

8 35	6	10	9	35
9 10	12 20	13 20	7	50
14	9	16 10	5 30	40
45	20	30	30	

(5)

We find the u_i's and v_j's by solving

$$u_1 = 0 \tag{6}$$

$$u_1 + v_1 = 8 \tag{7}$$

$$u_2 + v_1 = 9 \tag{8}$$

$$u_2 + v_2 = 12 \tag{9}$$

$$u_2 + v_3 = 13 \tag{10}$$

$$u_3 + v_3 = 16 \tag{11}$$

$$u_3 + v_4 = 5 \tag{12}$$

From (7), $v_1 = 8$. From (8), $u_2 = 1$. Then (9) yields $v_2 = 11$, and (10) yields $v_3 = 12$. From (11), $u_3 = 4$. Finally, (12) yields $v_4 = 1$. For each nonbasic variable, we now compute $\bar{c}_{ij} = u_i + v_j - c_{ij}$. We obtain

$$\bar{c}_{12} = 0 + 11 - 6 = 5 \qquad \bar{c}_{13} = 0 + 12 - 10 = 2$$

$$\bar{c}_{14} = 0 + 1 - 9 = -8 \qquad \bar{c}_{24} = 1 + 1 - 7 = -5$$

$$\bar{c}_{31} = 4 + 8 - 14 = -2 \qquad \bar{c}_{32} = 4 + 11 - 9 = 6$$

Because $\bar{c}_{32}$ is the most positive $\bar{c}_{ij}$, we would next enter x_{32} into the basis. Each unit of x_{32} that is entered into the basis will decrease Powerco's cost by \$6.

We can now summarize the procedure for using the transportation simplex to solve a transportation (min) problem.

Summary and Illustration of the Transportation Simplex Method

Step 1 If the problem is unbalanced, balance it.

Step 2 Use one of the methods described in Section 7.2 to find a bfs.

Step 3 Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the $[u_1 \quad u_2 \quad \ldots \quad u_m \quad v_1 \quad v_2 \quad \ldots \quad v_n]$ for the current bfs.

Step 4 If $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, then we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure. This yields a new bfs.

Step 5 Using the new bfs, return to steps 3 and 4.

For a maximization problem, proceed as stated, but replace step 4 by step 4′.

Step 4′ If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure described earlier.

We illustrate the procedure for solving a transportation problem by solving the Powerco problem. We begin with the bfs (5). We have already determined that x_{32} should enter the basis. As shown in Table 35, the loop involving x_{32} and some of the basic variables is (3, 2)–(3, 3)–(2, 3)–(2, 2). The odd cells in this loop are (3, 3) and (2, 2). Because $x_{33} = 10$ and $x_{22} = 20$, the pivot will decrease the value of x_{33} and x_{22} by 10 and increase the value of x_{32} and x_{23} by 10. The resulting bfs is shown in Table 36. The u_i's and v_j's for the new bfs were obtained by solving

$$\begin{array}{ll} u_1 = 0 & u_2 + v_3 = 13 \\ u_2 + v_2 = 12 & u_2 + v_1 = 9 \\ u_3 + v_4 = 5 & u_3 + v_2 = 9 \\ u_1 + v_1 = 8 & \end{array}$$

In computing $\bar{c}_{ij} = u_i + v_j - c_{ij}$ for each nonbasic variable, we find that $\bar{c}_{12} = 5$, $\bar{c}_{24} = 1$, and $\bar{c}_{13} = 2$ are the only positive $\bar{c}_{ij}$'s. Thus, we next enter x_{12} into the basis. The loop involving x_{12} and some of the basic variables is (1, 2)–(2, 2)–(2, 1)–(1, 1). The odd cells are (2, 2) and (1, 1). Because $x_{22} = 10$ is the smallest entry in an odd cell, we decrease x_{22} and x_{11} by 10 and increase x_{12} and x_{21} by 10. The resulting bfs is shown in Table 37. For this bfs, the u_i's and v_j's were determined by solving

$$\begin{array}{ll} u_1 = 0 & u_1 + v_2 = 6 \\ u_2 + v_1 = 9 & u_3 + v_2 = 9 \\ u_1 + v_1 = 8 & u_3 + v_4 = 5 \\ u_2 + v_3 = 13 & \end{array}$$

In computing $\bar{c}_{ij}$ for each nonbasic variable, we find that the only positive $\bar{c}_{ij}$ is $\bar{c}_{13} = 2$. Thus, x_{13} enters the basis. The loop involving x_{13} and some of the basic variables is

TABLE 35

Loop Involving Entering Variable x_{32}

8 35	6	10	9	35
9 10	12 20	13 20	7	50
14	9	16 10	5 30	40
45	20	30	30	

TABLE 36

x_{32} Has Entered the Basis, and x_{12} Enters Next

$v_j =$	8	11	12	7	
$u_i = 0$	8 35	6	10	9	35
1	9 10	12 10	13 30	7	50
−2	14	9 10	16	5 30	40
	45	20	30	30	

TABLE 37
x_{12} Has Entered the Basis, and x_{13} Enters Next

$v_j =$	8	6	12	2	
$u_i = 0$	8 25	6 10	10	9	35
1	9 20	12	13 30	7	50
3	14	9 10	16	5 30	40
	45	20	30	30	

TABLE 38
Optimal Tableau for Powerco

$v_j =$	6	6	10	2	
$u_i = 0$	8	6 10	10 25	9	35
3	9 45	12	13 5	7	50
3	14	9 10	16	5 30	40
	45	20	30	30	

(1, 3)–(2, 3)–(2, 1)–(1, 1). The odd cells are x_{23} and x_{11}. Because $x_{11} = 25$ is the smallest entry in an odd cell, we decrease x_{23} and x_{11} by 25 and increase x_{13} and x_{21} by 25. The resulting bfs is shown in Table 38. For this bfs, the u_i's and v_j's were obtained by solving

$$\begin{array}{rr} u_1 = 0 & u_2 + v_3 = 13 \\ u_2 + v_1 = 9 & u_1 + v_3 = 10 \\ u_3 + v_4 = 5 & u_3 + v_2 = 9 \\ u_1 + v_2 = 6 & \end{array}$$

The reader should check that for this bfs, all $\bar{c}_{ij} \leq 0$, so an optimal solution has been obtained. Thus, the optimal solution to the Powerco problem is $x_{12} = 10$, $x_{13} = 25$, $x_{21} = 45$, $x_{23} = 5$, $x_{32} = 10$, $x_{34} = 30$, and

$$z = 6(10) + 10(25) + 9(45) + 13(5) + 9(10) + 5(30) = \$1,020$$

PROBLEMS

Group A

Use the transportation simplex to solve Problems 1–8 in Section 7.1. Begin with the bfs found in Section 7.2.

7.4 Sensitivity Analysis for Transportation Problems[†]

We have already seen that for a transportation problem, the determination of a bfs and of row 0 for a given set of basic variables, as well as the pivoting procedure, all simplify. It should therefore be no surprise that certain aspects of the sensitivity analysis discussed in Section 6.3 can be simplified. In this section, we discuss the following three aspects of sensitivity analysis for the transportation problem:

Change 1 Changing the objective function coefficient of a nonbasic variable.

Change 2 Changing the objective function coefficient of a basic variable.

Change 3 Increasing a single supply by Δ and a single demand by Δ.

We illustrate three changes using the Powerco problem. Recall from Section 7.3 that the optimal solution for the Powerco problem was $z = \$1{,}020$; the optimal tableau is Table 39.

Changing the Objective Function Coefficient of a Nonbasic Variable

As in Section 6.3, changing the objective function coefficient of a nonbasic variable x_{ij} will leave the right-hand side of the optimal tableau unchanged. Thus, the current basis will still be feasible. We are not changing $\mathbf{c}_{\text{BV}}B^{-1}$, so the u_i's and v_j's remain unchanged. In row 0, only the coefficient of x_{ij} will change. Thus, as long as the coefficient of x_{ij} in the optimal row 0 is nonpositive, the current basis remains optimal.

To illustrate the method, we answer the following question: For what range of values of the cost of shipping 1 million kwh of electricity from plant 1 to city 1 will the current basis remain optimal? Suppose we change c_{11} from 8 to $8 + \Delta$. For what values of Δ will the current basis remain optimal? Now $\bar{c}_{11} = u_1 + v_1 - c_{11} = 0 + 6 - (8 + \Delta) = -2 - \Delta$. Thus, the current basis remains optimal for $-2 - \Delta \le 0$, or $\Delta \ge -2$, and $c_{11} \ge 8 - 2 = 6$.

Changing the Objective Function Coefficient of a Basic Variable

Because we are changing $\mathbf{c}_{\text{BV}}B^{-1}$, the coefficient of each nonbasic variable in row 0 may change, and to determine whether the current basis remains optimal, we must find the new u_i's and v_j's and use these values to price out all nonbasic variables. The current basis remains optimal as long as all nonbasic variables price out nonpositive. To illustrate the idea, we determine for the Powerco problem the range of values of the cost of shipping 1 million kwh from plant 1 to city 3 for which the current basis remains optimal.

Suppose we change c_{13} from 10 to $10 + \Delta$. Then the equation $\bar{c}_{13} = 0$ changes from $u_1 + v_3 = 10$ to $u_1 + v_3 = 10 + \Delta$. Thus, to find the u_i's and v_j's, we must solve the following equations:

$$\begin{aligned}
u_1 &= 0 & \qquad u_3 + v_2 &= 9\\
u_2 + v_1 &= 9 & u_1 + v_3 &= 10 + \Delta\\
u_1 + v_2 &= 6 & u_3 + v_4 &= 5\\
u_2 + v_3 &= 13 & &
\end{aligned}$$

[†] This section covers topics that may be omitted with no loss of continuity.

TABLE 39
Optimal Tableau for Powerco

		City 1	City 2	City 3	City 4	Supply
	$v_j =$	6	6	10	2	
Plant 1	$u_i = 0$	8	6 10	10 25	9	35
Plant 2	3	9 45	12	13 5	7	50
Plant 3	3	14	9 10	16	5 30	40
Demand		45	20	30	30	

Solving these equations, we obtain $u_1 = 0$, $v_2 = 6$, $v_3 = 10 + \Delta$, $v_1 = 6 + \Delta$, $u_2 = 3 - \Delta$, $u_3 = 3$, and $v_4 = 2$.

We now price out each nonbasic variable. The current basis will remain optimal as long as each nonbasic variable has a nonpositive coefficient in row 0.

$$\begin{aligned}
\bar{c}_{11} &= u_1 + v_1 - 8 = \Delta - 2 \le 0 && \text{for } \Delta \le 2 \\
\bar{c}_{14} &= u_1 + v_4 - 9 = -7 && \\
\bar{c}_{22} &= u_2 + v_2 - 12 = -3 - \Delta \le 0 && \text{for } \Delta \ge -3 \\
\bar{c}_{24} &= u_2 + v_4 - 7 = -2 - \Delta \le 0 && \text{for } \Delta \ge -2 \\
\bar{c}_{31} &= u_3 + v_1 - 14 = -5 + \Delta \le 0 && \text{for } \Delta \le 5 \\
\bar{c}_{33} &= u_3 + v_3 - 16 = \Delta - 3 \le 0 && \text{for } \Delta \le 3
\end{aligned}$$

Thus, the current basis remains optimal for $-2 \le \Delta \le 2$, or $8 = 10 - 2 \le c_{13} \le 10 + 2 = 12$.

Increasing Both Supply s_i and Demand d_j by Δ

Observe that this change maintains a balanced transportation problem. Because the u_i's and v_j's may be thought of as the negative of each constraint's shadow prices, we know from (37′) of Chapter 6 that if the current basis remains optimal,

$$\text{New } z\text{-value} = \text{old } z\text{-value} + \Delta u_i + \Delta v_j$$

For example, if we increase plant 1's supply and city 2's demand by 1 unit, then (new cost) = 1,020 + 1(0) + 1(6) = \$1,026.

We may also find the new values of the decision variables as follows:

1 If x_{ij} is a basic variable in the optimal solution, then increase x_{ij} by Δ.

2 If x_{ij} is a nonbasic variable in the optimal solution, then find the loop involving x_{ij} and some of the basic variables. Find an odd cell in the loop that is in row i. Increase the value of this odd cell by Δ and go around the loop, alternately increasing and then decreasing current basic variables in the loop by Δ.

To illustrate the first situation, suppose we increase s_1 and d_2 by 2. Because x_{12} is a basic variable in the optimal solution, the new optimal solution will be the one shown in Table 40. The new optimal z-value is $1,020 + 2u_1 + 2v_2 = \$1,032$. To illustrate the second situation, suppose we increase both s_1 and d_1 by 1. Because x_{11} is a nonbasic variable in the current optimal solution, we must find the loop involving x_{11} and some of the

TABLE 40

Optimal Tableau for Powerco If $s_1 = 35 + 2 = 37$ and $d_2 = 20 + 2 = 22$

		City 1	City 2	City 3	City 4	Supply
	$v_j =$	6	6	10	2	
Plant 1	$u_i = 0$	8	6 12	10 25	9	37
Plant 2	3	9 45	12	13 5	7	50
Plant 3	3	14	9 10	16	5 30	40
Demand		45	22	30	30	

TABLE 41

Optimal Tableau for Powerco If $s_1 = 35 + 1 = 36$ and $d_1 = 45 + 1 = 46$

		City 1	City 2	City 3	City 4	Supply
	$v_j =$	6	6	10	2	
Plant 1	$u_i = 0$	8	6 10	10 26	9	36
Plant 2	3	9 46	12	13 4	7	50
Plant 3	3	14	9 10	16	5 30	40
Demand		46	20	30	30	

basic variables. The loop is (1, 1)–(1, 3)–(2, 3)–(2, 1). The odd cell in the loop and row 1 is x_{13}. Thus, the new optimal solution will be obtained by increasing both x_{13} and x_{21} by 1 and decreasing x_{23} by 1. This yields the optimal solution shown in Table 41. The new optimal z-value is found from (new z-value) $= 1{,}020 + v_1 + v_1 = \$1{,}026$. Observe that if both s_1 and d_1 were increased by 6, the current basis would be infeasible. (Why?)

PROBLEMS

Group A

The following problems refer to the Powerco example.

1 Determine the range of values of c_{14} for which the current basis remains optimal.

2 Determine the range of values of c_{34} for which the current basis remains optimal.

3 If s_2 and d_3 are both increased by 3, what is the new optimal solution?

4 If s_3 and d_3 are both decreased by 2, what is the new optimal solution?

5 Two plants supply three customers with medical supplies. The unit costs of shipping from the plants to the customers, along with the supplies and demands, are given in Table 42.

a The company's goal is to minimize the cost of meeting customers' demands. Find two optimal bfs for this transportation problem.

b Suppose that customer 2's demand increased by one unit. By how much would costs increase?

TABLE 42

	To			
From	Customer 1	Customer 2	Customer 3	Supply
Plant 1	\$55	\$65	\$80	35
Plant 2	\$10	\$15	\$25	50
Demand	10	10	10	

7.5 Assignment Problems

Although the transportation simplex appears to be very efficient, there is a certain class of transportation problems, called assignment problems, for which the transportation simplex is often very inefficient. In this section, we define assignment problems and discuss an efficient method that can be used to solve them.

EXAMPLE 4 Machine Assignment Problem

Machineco has four machines and four jobs to be completed. Each machine must be assigned to complete one job. The time required to set up each machine for completing each job is shown in Table 43. Machineco wants to minimize the total setup time needed to complete the four jobs. Use linear programming to solve this problem.

Solution Machineco must determine which machine should be assigned to each job. We define (for $i, j = 1, 2, 3, 4$)

$$x_{ij} = 1 \text{ if machine } i \text{ is assigned to meet the demands of job } j$$
$$x_{ij} = 0 \text{ if machine } i \text{ is not assigned to meet the demands of job } j$$

Then Machineco's problem may be formulated as

$$\begin{aligned}
\min z = {} & 14x_{11} + 5x_{12} + 8x_{13} + 7x_{14} + 2x_{21} + 12x_{22} + 6x_{23} + 5x_{24} \\
& + 7x_{31} + 8x_{32} + 3x_{33} + 9x_{34} + 2x_{41} + 4x_{42} + 6x_{43} + 10x_{44} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} + x_{14} = 1 \qquad \text{(Machine constraints)} \\
& x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 1 \\
& x_{41} + x_{42} + x_{43} + x_{44} = 1 \\
& x_{11} + x_{21} + x_{31} + x_{41} = 1 \qquad \text{(Job constraints)} \\
& x_{12} + x_{22} + x_{32} + x_{42} = 1 \\
& x_{13} + x_{23} + x_{33} + x_{43} = 1 \\
& x_{14} + x_{24} + x_{34} + x_{44} = 1 \\
& x_{ij} = 0 \quad \text{or} \quad x_{ij} = 1
\end{aligned} \tag{13}$$

The first four constraints in (13) ensure that each machine is assigned to a job, and the last four ensure that each job is completed. If $x_{ij} = 1$, then the objective function will pick up the time required to set up machine i for job j; if $x_{ij} = 0$, then the objective function will not pick up the time required.

Ignoring for the moment the $x_{ij} = 0$ or $x_{ij} = 1$ restrictions, we see that Machineco faces a balanced transportation problem in which each supply point has a supply of 1 and each

TABLE 43
Setup Times for Machineco

	Time (Hours)			
Machine	Job 1	Job 2	Job 3	Job 4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

demand point has a demand of 1. In general, an **assignment problem** is a balanced transportation problem in which all supplies and demands are equal to 1. Thus, an assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point. The assignment problem’s matrix of costs is its **cost matrix.**

All the supplies and demands for the Machineco problem (and for any assignment problem) are integers, so our discussion in Section 7.3 implies that all variables in Machineco’s optimal solution must be integers. Because the right-hand side of each constraint is equal to 1, each x_{ij} must be a nonnegative integer that is no larger than 1, so each x_{ij} must equal 0 or 1. This means that we can ignore the restrictions that $x_{ij} = 0$ or 1 and solve (13) as a balanced transportation problem. By the minimum cost method, we obtain the bfs in Table 44. The current bfs is highly degenerate. (In any bfs to an $m \times m$ assignment problem, there will always be m basic variables that equal 1 and $m - 1$ basic variables that equal 0.)

We find that $\bar{c}_{43} = 1$ is the only positive $\bar{c}_{ij}$. We therefore enter x_{43} into the basis. The loop involving x_{43} and some of the basic variables is (4, 3)–(1, 3)–(1, 2)–(4, 2). The odd variables in the loop are x_{13} and x_{42}. Because $x_{13} = x_{42} = 0$, either x_{13} or x_{42} will leave

TABLE 44
Basic Feasible Solution for Machineco

		Job 1	Job 2	Job 3	Job 4	
	$v_j =$	3	4	8	7	
Machine 1	$u_i = 0$	$\boxed{14}$	$\boxed{5}$ 1	$\boxed{8}$ 0	$\boxed{7}$ 0	1
Machine 2	−2	$\boxed{2}$	$\boxed{12}$	$\boxed{6}$	$\boxed{5}$ 1	1
Machine 3	−5	$\boxed{7}$	$\boxed{8}$	$\boxed{3}$ 1	$\boxed{9}$	1
Machine 4	−1	$\boxed{2}$ 1	$\boxed{4}$ 0	$\boxed{6}$	$\boxed{10}$	1
		1	1	1	1	

TABLE 45
x_{43} Has Entered the Basis

		Job 1	Job 2	Job 3	Job 4	
	$v_j =$	3	5	7	7	
Machine 1	$u_i = 0$	$\boxed{14}$	$\boxed{5}$ 1	$\boxed{8}$	$\boxed{7}$ 0	1
Machine 2	−2	$\boxed{2}$	$\boxed{12}$	$\boxed{6}$	$\boxed{5}$ 1	1
Machine 3	−4	$\boxed{7}$	$\boxed{8}$	$\boxed{3}$ 1	$\boxed{9}$	1
Machine 4	−1	$\boxed{2}$ 1	$\boxed{4}$ 0	$\boxed{6}$ 0	$\boxed{10}$	1
		1	1	1	1	

the basis. We arbitrarily choose x_{13} to leave the basis. After performing the pivot, we obtain the bfs in Table 45. All $\bar{c}_{ij}$'s are now nonpositive, so we have obtained an optimal assignment: $x_{12} = 1$, $x_{24} = 1$, $x_{33} = 1$, and $x_{41} = 1$. Thus, machine 1 is assigned to job 2, machine 2 is assigned to job 4, machine 3 is assigned to job 3, and machine 4 is assigned to job 1. A total setup time of $5 + 5 + 3 + 2 = 15$ hours is required.

The Hungarian Method

Looking back at our initial bfs, we see that it was an optimal solution. We did not know that it was optimal, however, until performing one iteration of the transportation simplex. This suggests that the high degree of degeneracy in an assignment problem may cause the transportation simplex to be an inefficient way of solving assignment problems. For this reason (and the fact that the algorithm is even simpler than the transportation simplex), the Hungarian method is usually used to solve assignment (min) problems:

Step 1 Find the minimum element in each row of the $m \times m$ cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (called the reduced cost matrix) by subtracting from each cost the minimum cost in its column.

Step 2 Draw the minimum number of lines (horizontal, vertical, or both) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, then an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, then proceed to step 3.

Step 3 Find the smallest nonzero element (call its value k) in the reduced cost matrix that is uncovered by the lines drawn in step 2. Now subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to step 2.

REMARKS

1 To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by -1 and solve the problem as a minimization problem.

2 If the number of rows and columns in the cost matrix are unequal, then the assignment problem is unbalanced. The Hungarian method may yield an incorrect solution if the problem is unbalanced. Thus, any assignment problem should be balanced (by the addition of one or more dummy points) before it is solved by the Hungarian method.

3 In a large problem, it may not be easy to find the minimum number of lines needed to cover all zeros in the current cost matrix. For a discussion of how to find the minimum number of lines needed, see Gillett (1976). It can be shown that if j lines are required, then only j "jobs" can be assigned to zero costs in the current matrix. This explains why the algorithm terminates when m lines are required.

Solution of Machineco Example by the Hungarian Method

We illustrate the Hungarian method by solving the Machineco problem (see Table 46).

Step 1 For each row, we subtract the row minimum from each element in the row, obtaining Table 47. We now subtract 2 from each cost in column 4, obtaining Table 48.

Step 2 As shown, lines through row 1, row 3, and column 1 cover all the zeros in the reduced cost matrix. From remark 3, it follows that only three jobs can be assigned to zero costs in the current cost matrix. Fewer than four lines are required to cover all the zeros, so we proceed to step 3.

TABLE 46
Cost Matrix for Machineco

				Row Minimum
14	5	8	7	5
2	12	6	5	2
7	8	3	9	3
2	4	6	10	2

TABLE 47
Cost Matrix After Row Minimums Are Subtracted

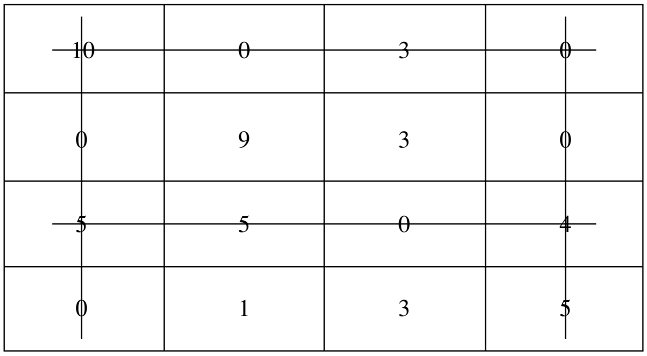
9	0	3	2
0	10	4	3
4	5	0	6
0	2	4	8
Column Minimum	0	0	2

TABLE 48
Cost Matrix After Column Minimums Are Subtracted

~~9~~	~~0~~	~~3~~	~~0~~
~~0~~	10	4	1
~~4~~	~~5~~	~~0~~	~~4~~
~~0~~	2	4	6

Step 3 The smallest uncovered element equals 1, so we now subtract 1 from each uncovered element in the reduced cost matrix and add 1 to each twice-covered element. The resulting matrix is Table 49. Four lines are now required to cover all the zeros. Thus, an optimal solution is available. To find an optimal assignment, observe that the only covered 0 in column 3 is x_{33}, so we must have $x_{33} = 1$. Also, the only available covered zero in column 2 is x_{12}, so we set $x_{12} = 1$ and observe that neither row 1 nor column 2 can be used again. Now the only available covered zero in column 4 is x_{24}. Thus, we choose $x_{24} = 1$ (which now excludes both row 2 and column 4 from further use). Finally, we choose $x_{41} = 1$.

TABLE 49
Four Lines Required; Optimal Solution Is Available



10	0	3	0
0	9	3	0
5	5	0	4
0	1	3	5

Thus, we have found the optimal assignment $x_{12} = 1$, $x_{24} = 1$, $x_{33} = 1$, and $x_{41} = 1$. Of course, this agrees with the result obtained by the transportation simplex.

Intuitive Justification of the Hungarian Method

To give an intuitive explanation of why the Hungarian algorithm works, we need to discuss the following result: *If a constant is added to each cost in a row (or column) of a balanced transportation problem, then the optimal solution to the problem is unchanged.* To show why the result is true, suppose we add k to each cost in the first row of the Machineco problem. Then

$$\text{New objective function} = \text{old objective function} + k(x_{11} + x_{12} + x_{13} + x_{14})$$

Because any feasible solution to the Machineco problem must have $x_{11} + x_{12} + x_{13} + x_{14} = 1$,

$$\text{New objective function} = \text{old objective function} + k$$

Thus, the optimal solution to the Machineco problem remains unchanged if a constant k is added to each cost in the first row. A similar argument applies to any other row or column.

Step 1 of the Hungarian method consists (for each row and column) of subtracting a constant from each element in the row or column. Thus, step 1 creates a new cost matrix having the same optimal solution as the original problem. Step 3 of the Hungarian method is equivalent (see Problem 7 at the end of this section) to adding k to each cost that lies in a covered row and subtracting k from each cost that lies in an uncovered column (or vice versa). Thus, step 3 creates a new cost matrix with the same optimal solution as the initial assignment problem. Each time step 3 is performed, at least one new zero is created in the cost matrix.

Steps 1 and 3 also ensure that all costs remain nonnegative. Thus, the net effect of steps 1 and 3 of the Hungarian method is to create a sequence of assignment problems (with nonnegative costs) that all have the same optimal solution as the original assignment problem. Now consider an assignment problem in which all costs are nonnegative. Any feasible assignment in which all the x_{ij}'s that equal 1 have zero costs must be optimal for such an assignment problem. Thus, when step 2 indicates that m lines are required to cover all the zeros in the cost matrix, an optimal solution to the original problem has been found.

Computer Solution of Assignment Problems

To solve assignment problems in LINDO, type in the objective function and constraints. Also, many menu-driven programs require the user to input only a list of supply and demand points (such as jobs and machines, respectively) and a cost matrix. LINGO can also be used to easily solve assignment problems, including the following model to solve the Machineco example (file Assign.lng).

Assign.lng

```
MODEL:
   1]SETS:
   2]MACHINES/1..4/;
   3]JOBS/1..4/;
   4]LINKS(MACHINES,JOBS):COST,ASSIGN;
   5]ENDSETS
   6]MIN=@SUM(LINKS:COST*ASSIGN);
   7]@FOR(MACHINES(I):
   8]@SUM(JOBS(J):ASSIGN(I,J))<1);
   9]@FOR(JOBS(J):
  10]@SUM(MACHINES(I):ASSIGN(I,J))>1);
  11]DATA:
  12]COST = 14,5,8,7,
  13]2,12,6,5,
  14]7,8,3,9,
  15]2,4,6,10;
  16]ENDDATA
END
```

Line 2 defines the four supply points (machines), and line 3 defines the four demand points (jobs). In line 4, we define each possible combination of jobs and machines (16 in all) and associate with each combination an assignment cost [for example COST(1, 2) = 5] and a variable ASSIGN(I,J). ASSIGN(I,J) equals 1 if machine *i* is used to perform job *j*; it equals 0 otherwise. Line 5 ends the definition of sets.

Line 6 expresses the objective function by summing over all possible (I,J) combinations the product of the assignment cost and ASSIGN(I,J). Lines 7–8 limit each MACHINE to performing at most one job by forcing (for each machine) the sum of ASSIGN(I,J) over all JOBS to be at most 1. Lines 9–10 require that each JOB be completed by forcing (for each job) the sum of ASSIGN(I,J) over all MACHINES to be at least 1.

Lines 12–16 input the cost matrix.

Observe that this LINGO program can (with simple editing) be used to solve any assignment problem (even if it is not balanced!). For example, if you had 10 machines available to perform 8 jobs, you would edit line 2 to indicate that there are 10 machines (replace 1..4 with 1..10). Then edit line 3 to indicate that there are 8 jobs. Finally, in line 12, you would type the 80 entries of your cost matrix, following "COST=" and you would be ready to roll!

REMARK **1** From our discussion of the Machineco example, it is unnecessary to force the ASSIGN(I,J) to equal 0 or 1; this will happen automatically!

PROBLEMS

Group A

1 Five employees are available to perform four jobs. The time it takes each person to perform each job is given in Table 50. Determine the assignment of employees to jobs that minimizes the total time required to perform the four jobs.

TABLE 50

	Time (hours)			
Person	**Job 1**	**Job 2**	**Job 3**	**Job 4**
1	22	18	30	18
2	18	—	27	22
3	26	20	28	28
4	16	22	—	14
5	21	—	25	28

Note: Dashes indicate person cannot do that particular job.

2[†] Doc Councillman is putting together a relay team for the 400-meter relay. Each swimmer must swim 100 meters of breaststroke, backstroke, butterfly, or freestyle. Doc believes that each swimmer will attain the times given in

---

[†] This problem is based on Machol (1970).

Table 51. To minimize the team's time for the race, which swimmer should swim which stroke?

3 Tom Cruise, Freddy Prinze Jr., Harrison Ford, and Matt LeBlanc are marooned on a desert island with Jennifer Aniston, Courteney Cox, Gwyneth Paltrow, and Julia Roberts. The "compatibility measures" in Table 52 indicate how much happiness each couple would experience if they spent all their time together. The happiness earned by a couple is proportional to the fraction of time they spend together. For example, if Freddie and Gwyneth spend half their time together, they earn happiness of $\frac{1}{2}(9) = 4.5$.

- **a** Let x_{ij} be the fraction of time that the ith man spends with the jth woman. The goal of the eight people is to maximize the total happiness of the people on the island. Formulate an LP whose optimal solution will yield the optimal values of the x_{ij}'s.
- **b** Explain why the optimal solution in part (a) will have four $x_{ij} = 1$ and twelve $x_{ij} = 0$. The optimal solution requires that each person spend all his or her time with one person of the opposite sex, so this result is often referred to as the Marriage Theorem.
- **c** Determine the marriage partner for each person.
- **d** Do you think the Proportionality Assumption of linear programming is valid in this situation?

4 A company is taking bids on four construction jobs. Three people have placed bids on the jobs. Their bids (in thousands of dollars) are given in Table 53 (a * indicates that the person did not bid on the given job). Person 1 can do only one job, but persons 2 and 3 can each do as many as two jobs. Determine the minimum cost assignment of persons to jobs.

5 Greydog Bus Company operates buses between Boston and Washington, D.C. A bus trip between these two cities takes 6 hours. Federal law requires that a driver rest for four or more hours between trips. A driver's workday consists of two trips: one from Boston to Washington and one from Washington to Boston. Table 54 gives the departure times for the buses. Greydog's goal is to minimize the total downtime for all drivers. How should Greydog assign crews to trips? *Note:* It is permissible for a driver's "day" to overlap midnight. For example, a Washington-based driver can be assigned to the Washington–Boston 3 P.M. trip and the Boston–Washington 6 A.M. trip.

6 Five male characters (Billie, John, Fish, Glen, and Larry) and five female characters (Ally, Georgia, Jane, Rene, and Nell) from *Ally McBeal* are marooned on a desert island. The problem is to determine what percentage of time each woman on the island should spend with each man. For example, Ally could spend 100% of her time with John or she could "play the field" by spending 20% of her time with each man. Table 55 shows a "happiness index" for each potential pairing of a man and woman. For example, if Larry and Rene spend all their time together, they earn 8 units of happiness for the island.

- **a** Play matchmaker and determine an allocation of each man and woman's time that earns the maximum total happiness for the island. Assume that happiness earned by a couple is proportional to the amount of time they spend together.
- **b** Explain why the optimal solution to this problem will, for any matrix of "happiness indices," always involve each woman spending all her time with one man.

TABLE 51

	Time (seconds)			
Swimmer	**Free**	**Breast**	**Fly**	**Back**
Gary Hall	54	54	51	53
Mark Spitz	51	57	52	52
Jim Montgomery	50	53	54	56
Chet Jastremski	56	54	55	53

TABLE 52

	JA	CC	GP	JR
TC	7	5	8	2
FP	7	8	9	4
HF	3	5	7	9
ML	5	5	6	7

TABLE 53

	Job			
Person	**1**	**2**	**3**	**4**
1	50	46	42	40
2	51	48	44	*
3	*	47	45	45

TABLE 54

Trip	Departure Time	Trip	Departure Time
Boston 1	6 A.M.	Washington 1	5:30 A.M.
Boston 2	7:30 A.M.	Washington 2	9 A.M.
Boston 3	11.30 A.M.	Washington 3	3 P.M.
Boston 4	7 P.M.	Washington 4	6:30 P.M.
Boston 5	12:30 A.M.	Washington 5	12 midnight

TABLE 55

	Ally	Georgia	Jane	Rene	Nell
Billie	8	6	4	7	5
John	5	7	6	4	9
Fish	10	6	5	2	10
Glen	1	0	0	0	0
Larry	5	7	9	8	6

c What assumption made in the problem is needed for the Marriage Theorem to hold?

Group B

7 Any transportation problem can be formulated as an assignment problem. To illustrate the idea, determine an assignment problem that could be used to find the optimal solution to the transportation problem in Table 56. (*Hint:* You will need five supply and five demand points).

8 The Chicago board of education is taking bids on the city's four school bus routes. Four companies have made the bids in Table 57.

a Suppose each bidder can be assigned only one route. Use the assignment method to minimize Chicago's cost of running the four bus routes.

TABLE 56

3	1	2
2	3	3
1	4	

TABLE 57

	Bids			
Company	Route 1	Route 2	Route 3	Route 4
1	$4,000	$5,000	—	—
2	—	$4,000	—	$4,000
3	$3,000	—	$2,000	—
4	—	—	$4,000	$5,000

b Suppose that each company can be assigned two routes. Use the assignment method to minimize Chicago's cost of running the four bus routes. (*Hint:* Two supply points will be needed for each company.)

9 Show that step 3 of the Hungarian method is equivalent to performing the following operations: (1) Add k to each cost that lies in a covered row. (2) Subtract k from each cost that lies in an uncovered column.

10 Suppose c_{ij} is the smallest cost in row i and column j of an assignment problem. Must $x_{ij} = 1$ in any optimal assignment?

7.6 Transshipment Problems

A transportation problem allows only shipments that go directly from a supply point to a demand point. In many situations, shipments are allowed between supply points or between demand points. Sometimes there may also be points (called *transshipment points*) through which goods can be transshipped on their journey from a supply point to a demand point. Shipping problems with any or all of these characteristics are transshipment problems. Fortunately, the optimal solution to a transshipment problem can be found by solving a transportation problem.

In what follows, we define a **supply point** to be a point that can send goods to another point but cannot receive goods from any other point. Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other point. A **transshipment point** is a point that can both receive goods from other points and send goods to other points. The following example illustrates these definitions ("—" indicates that a shipment is impossible).

EXAMPLE 5 Transshipment

Widgetco manufactures widgets at two factories, one in Memphis and one in Denver. The Memphis factory can produce as many as 150 widgets per day, and the Denver factory can produce as many as 200 widgets per day. Widgets are shipped by air to customers in Los Angeles and Boston. The customers in each city require 130 widgets per day. Because of the deregulation of airfares, Widgetco believes that it may be cheaper to first fly some widgets to New York or Chicago and then fly them to their final destinations. The costs of flying a widget are shown in Table 58. Widgetco wants to minimize the total cost of shipping the required widgets to its customers.

TABLE 58

Shipping Costs for Transshipments

	To ($)					
From	**Memphis**	**Denver**	**N.Y.**	**Chicago**	**L.A.**	**Boston**
Memphis	0	—	8	13	25	28
Denver	—	0	15	12	26	25
N.Y.	—	—	0	6	16	17
Chicago	—	—	6	0	14	16
L.A.	—	—	—	—	0	—
Boston	—	—	—	—	—	0

In this problem, Memphis and Denver are supply points, with supplies of 150 and 200 widgets per day, respectively. New York and Chicago are transshipment points. Los Angeles and Boston are demand points, each with a demand of 130 widgets per day. A graphical representation of possible shipments is given in Figure 9.

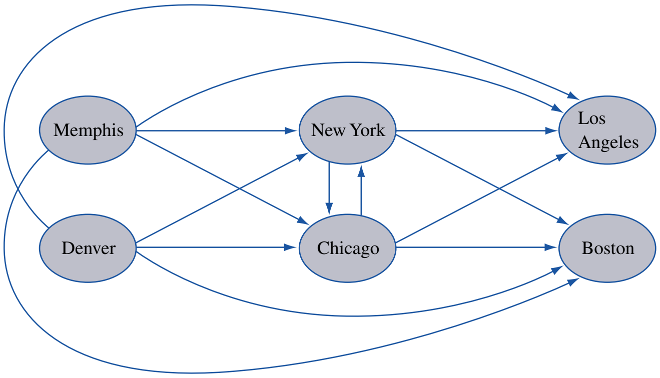
We now describe how the optimal solution to a transshipment problem can be found by solving a transportation problem. Given a transshipment problem, we create a balanced transportation problem by the following procedure (assume that total supply exceeds total demand):

Step 1 If necessary, add a dummy demand point (with a supply of 0 and a demand equal to the problem's excess supply) to balance the problem. Shipments to the dummy and from a point to itself will, of course, have a zero shipping cost. Let s = total available supply.

Step 2 Construct a transportation tableau as follows: A row in the tableau will be needed for each supply point and transshipment point, and a column will be needed for each demand point and transshipment point. Each supply point will have a supply equal to its original supply, and each demand point will have a demand equal to its original demand. Let s = total available supply. Then each transshipment point will have a supply equal to (point's original supply) $+ s$ and a demand equal to (point's original demand) $+ s$. This ensures that any transshipment point that is a net supplier will have a net outflow equal to the point's original supply, and, similarly, a net demander will have a net inflow equal to the point's original demand. Although we don't know how much will be shipped through each transshipment point, we can be sure that the total amount will not exceed s. This explains why we add s to the supply and demand at each transshipment point. By adding the same amounts to the supply and demand, we ensure that the net outflow at each transshipment point will be correct, and we also maintain a balanced transportation tableau.

FIGURE 9

A Transshipment Problem



For the Widgetco example, this procedure yields the transportation tableau and its optimal solution given in Table 59. Because s = (total supply) = 150 + 200 = 350 and (total demand) = 130 + 130 = 260, the dummy demand point has a demand of 350 − 260 = 90. The other supplies and demands in the transportation tableau are obtained by adding $s = 350$ to each transshipment point's supply and demand.

In interpreting the solution to the transportation problem created from a transshipment problem, we simply ignore the shipments to the dummy and from a point to itself. From Table 59, we find that Widgetco should produce 130 widgets at Memphis, ship them to New York, and transship them from New York to Los Angeles. The 130 widgets produced at Denver should be shipped directly to Boston. The net outflow from each city is

$$\begin{aligned}
&\text{Memphis:} && 130 + 20 = 150\\
&\text{Denver:} && 130 + 70 = 200\\
&\text{N.Y.:} && 220 + 130 - 130 - 220 = 0\\
&\text{Chicago:} && 350 - 350 = 0\\
&\text{L.A.:} && -130\\
&\text{Boston:} && -130\\
&\text{Dummy:} && -20 - 70 = -90
\end{aligned}$$

A negative net outflow represents an inflow. Observe that each transshipment point (New York and Chicago) has a net outflow of 0; whatever flows into the transshipment point must leave the transshipment point. A graphical representation of the optimal solution to the Widgetco example is given in Figure 10.

Suppose that we modify the Widgetco example and allow shipments between Memphis and Denver. This would make Memphis and Denver transshipment points and would add columns for Memphis and Denver to the Table 59 tableau. The Memphis row in the tableau would now have a supply of 150 + 350 = 500, and the Denver row would have

TABLE 59
Representation of Transshipment Problem as Balanced Transportation Problem

	N.Y.	**Chicago**	**L.A.**	**Boston**	**Dummy**	**Supply**
Memphis	8 130	13	25	28	0 20	150
Denver	15	12	26	25 130	0 70	200
N.Y.	0 220	6	16 130	17	0	350
Chicago	6	0 350	14	16	0	350
Demand	350	350	130	130	90	

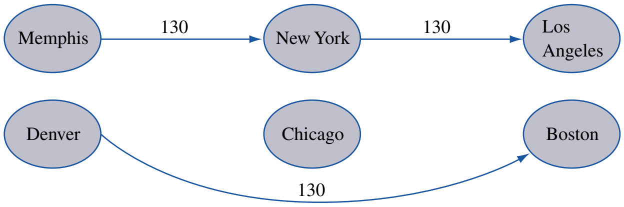


FIGURE 10
Optimal Solution to Widgetco

a supply of 200 + 350 = 550. The new Memphis column would have a demand of 0 + 350 = 350, and the new Denver column would have a demand of 0 + 350 = 350. Finally, suppose that shipments between demand points L.A. and Boston were allowed. This would make L.A. and Boston transshipment points and add rows for L.A. and Boston. The supply for both the L.A. and Boston rows would be 0 + 350 = 350. The demand for both the L.A. and Boston columns would now be 130 + 350 = 480.

PROBLEMS

Group A

1 General Ford produces cars at L.A. and Detroit and has a warehouse in Atlanta; the company supplies cars to customers in Houston and Tampa. The cost of shipping a car between points is given in Table 60 ("—" means that a shipment is not allowed). L.A. can produce as many as 1,100 cars, and Detroit can produce as many as 2,900 cars. Houston must receive 2,400 cars, and Tampa must receive 1,500 cars.

- **a** Formulate a balanced transportation problem that can be used to minimize the shipping costs incurred in meeting demands at Houston and Tampa.
- **b** Modify the answer to part (a) if shipments between L.A. and Detroit are not allowed.
- **c** Modify the answer to part (a) if shipments between Houston and Tampa are allowed at a cost of $5.

2 Sunco Oil produces oil at two wells. Well 1 can produce as many as 150,000 barrels per day, and well 2 can produce as many as 200,000 barrels per day. It is possible to ship oil directly from the wells to Sunco's customers in Los Angeles and New York. Alternatively, Sunco could transport oil to the ports of Mobile and Galveston and then ship it by tanker to New York or Los Angeles. Los Angeles requires 160,000 barrels per day, and New York requires 140,000 barrels per day. The costs of shipping 1,000 barrels between two points are shown in Table 61. Formulate a transshipment model (and equivalent transportation model) that could be used to minimize the transport costs in meeting the oil demands of Los Angeles and New York.

3 In Problem 2, assume that before being shipped to Los Angeles or New York, all oil produced at the wells must be refined at either Galveston or Mobile. To refine 1,000 barrels of oil costs $12 at Mobile and $10 at Galveston. Assuming that both Mobile and Galveston have infinite refinery capacity, formulate a transshipment and balanced transportation model to minimize the daily cost of transporting and refining the oil requirements of Los Angeles and New York.

TABLE 60

	To ($)				
From	**L.A.**	**Detroit**	**Atlanta**	**Houston**	**Tampa**
L.A.	0	140	100	90	225
Detroit	145	0	111	110	119
Atlanta	105	115	0	113	78
Houston	89	109	121	0	—
Tampa	210	117	82	—	0

TABLE 61

	To ($)					
From	**Well 1**	**Well 2**	**Mobile**	**Galveston**	**N.Y.**	**L.A.**
Well 1	0	—	10	13	25	28
Well 2	—	0	15	12	26	25
Mobile	—	—	0	6	16	17
Galveston	—	—	6	0	14	16
N.Y.	—	—	—	—	0	15
L.A.	—	—	—	—	15	0

Note: Dashes indicate shipments that are not allowed.

4 Rework Problem 3 under the assumption that Galveston has a refinery capacity of 150,000 barrels per day and Mobile has one of 180,000 barrels per day. (*Hint:* Modify the method used to determine the supply and demand at each transshipment point to incorporate the refinery capacity restrictions, but make sure to keep the problem balanced.)

5 General Ford has two plants, two warehouses, and three customers. The locations of these are as follows:

Plants: Detroit and Atlanta
Warehouses: Denver and New York
Customers: Los Angeles, Chicago, and Philadelphia

Cars are produced at plants, then shipped to warehouses, and finally shipped to customers. Detroit can produce 150 cars per week, and Atlanta can produce 100 cars per week. Los Angeles requires 80 cars per week; Chicago, 70; and Philadelphia, 60. It costs $10,000 to produce a car at each plant, and the cost of shipping a car between two cities is given in Table 62. Determine how to meet General Ford's weekly demands at minimum cost.

Group B

6[†] A company must meet the following demands for cash at the beginning of each of the next six months: month 1,

---

[†] Based on Srinivasan (1974).

TABLE 62

	To ($)	
From	Denver	New York
Detroit	1,253	637
Atlanta	1,398	841

	To ($)		
From	Los Angeles	Chicago	Philadelphia
Denver	1,059	996	1,691
New York	2,786	802	100

TABLE 63

	Month of Sale					
Bond	1	2	3	4	5	6
1	$0.21	$0.19	$0.17	$0.13	$0.09	$0.05
2	$0.50	$0.50	$0.50	$0.33	$0.00	$0.00
3	$1.00	$1.00	$1.00	$1.00	$1.00	$0.00

$200; month 2, $100; month 3, $50; month 4, $80; month 5, $160; month 6, $140. At the beginning of month 1, the company has $150 in cash and $200 worth of bond 1, $100 worth of bond 2, and $400 worth of bond 3. The company will have to sell some bonds to meet demands, but a penalty will be charged for any bonds sold before the end of month 6. The penalties for selling $1 worth of each bond are as shown in Table 63.

a Assuming that all bills must be paid on time, formulate a balanced transportation problem that can be used to minimize the cost of meeting the cash demands for the next six months.

b Assume that payment of bills can be made after they are due, but a penalty of 5¢ per month is assessed for each dollar of cash demands that is postponed for one month. Assuming all bills must be paid by the end of month 6, develop a transshipment model that can be used to minimize the cost of paying the next six months' bills. (*Hint:* Transshipment points are needed, in the form Ct = cash available at beginning of month t after bonds for month t have been sold, but before month t demand is met. Shipments into Ct occur from bond sales and $Ct - 1$. Shipments out of Ct occur to $Ct + 1$ and demands for months 1, 2, ... t.)

SUMMARY

Notation

m = number of supply points

n = number of demand points

x_{ij} = number of units shipped from supply point i to demand point j

c_{ij} = cost of shipping 1 unit from supply point i to demand point j

s_i = supply at supply point i

d_j = demand at demand point j

$\bar{c}_{ij}$ = coefficient of x_{ij} in row 0 of a given tableau

$\mathbf{a}_{ij}$ = column for x_{ij} in transportation constraints

A transportation problem is **balanced** if total supply equals total demand. To use the methods of this chapter to solve a transportation problem, the problem must first be balanced by use of a dummy supply or a dummy demand point. A balanced transportation problem may be written as

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} = s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)}$$

$$\text{s.t.} \quad \sum_{i=1}^{i=m} x_{ij} = d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)}$$

$$x_{ij} \geq 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$$

Finding Basic Feasible Solutions for Balanced Transportation Problems

We can find a bfs for a balanced transportation problem by the northwest corner method, the minimum-cost method, or Vogel's method. To find a bfs by the northwest corner method, begin in the upper left-hand (or northwest) corner of the transportation tableau and set x_{11} as large as possible. Clearly, x_{11} can be no larger than the smaller of s_1 and d_1. If $x_{11} = s_1$, then cross out the first row of the transportation tableau; this indicates that no more basic variables will come from row 1 of the tableau. Also change d_1 to $d_1 - s_1$. If $x_{11} = d_1$, then cross out the first column of the transportation tableau and change s_1 to $s_1 - d_1$. If $x_{11} = s_1 = d_1$, cross out either row 1 or column 1 (but not both) of the transportation tableau. If you cross out row 1, change d_1 to 0; if you cross out column 1, change s_1 to 0. Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed-out row or column. Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row and its column. A basic feasible solution has now been obtained.

Finding the Optimal Solution for a Transportation Problem

Step 1 If the problem is unbalanced, balance it.

Step 2 Use one of the methods described in Section 7.2 to find a bfs.

Step 3 Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the $[u_1 \quad u_2 \ldots u_m \quad v_1 \quad v_2 \ldots v_n]$ for the current bfs.

Step 4 If $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, then we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis. To do this, find the loop. Then, *counting only cells in the loop,* label the even cells. Also label the odd cells. Now find the odd cell whose variable assumes the smallest value, θ. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by θ and increase the value of each even cell by θ. The values of variables not in the loop remain unchanged. The pivot is now complete. If $\theta = 0$, then the entering variable will equal 0, and an odd variable that has a current value of 0 will leave the basis. In this case, a degenerate bfs will result. If more than one odd cell in the loop equals θ, you may arbitrarily choose one of these odd cells to leave the basis; again, a degenerate bfs will result. The pivoting yields a new bfs.

Step 5 Using the new bfs, return to steps 3 and 4.

For a maximization problem, proceed as stated, but replace step 4 by step 4′.

Step 4′ If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure.

Assignment Problems

An **assignment problem** is a balanced transportation problem in which all supplies and demands equal 1. An $m \times m$ assignment problem may be efficiently solved by the Hungarian method:

Step 1 Find the minimum element in each row of the cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (reduced cost matrix) by subtracting from each cost the minimum cost in its column.

Step 2 Cover all the zeros in the reduced cost matrix using the minimum number of lines needed. If m lines are required, then an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, then proceed to step 3.

Step 3 Find the smallest nonzero element (k) in the reduced cost matrix that is uncovered by the lines drawn in step 2. Now subtract k from each uncovered element and add k to each element that is covered by two lines. Return to step 2.

REMARKS **1** To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by -1 and solve it as a minimization problem.
2 If the number of rows and columns in the cost matrix are unequal, then the problem is unbalanced. The Hungarian method may yield an incorrect solution if the problem is unbalanced. Thus, any assignment problem should be balanced (by the addition of one or more dummy points) before it is solved by the Hungarian method.

Transshipment Problems

A transshipment problem allows shipment between supply points and between demand points, and it may also contain transshipment points through which goods may be shipped on their way from a supply point to a demand point. Using the following method, a transshipment problem may be transformed into a balanced transportation problem.

Step 1 If necessary, add a dummy demand point (with a supply of 0 and a demand equal to the problem's excess supply) to balance the problem. Shipments to the dummy and from a point to itself will, of course, have a zero shipping cost. Let s = total available supply.

Step 2 Construct a transportation tableau creating a row for each supply point and transshipment point, and a column for each demand point and transshipment point. Each supply point will have a supply equal to its original supply, and each demand point will have a demand equal to its original demand. Let s = total available supply. Then each transshipment point will have a supply equal to (point's original supply) + s and a demand equal to (point's original demand) + s.

Sensitivity Analysis for Transportation Problems

Following the discussion of sensitivity analysis in Chapter 6, we can analyze how a change in a transportation problem affects the problem's optimal solution.

Change 1 Changing the objective function coefficient of a nonbasic variable. As long as the coefficient of x_{ij} in the optimal row 0 is nonpositive, the current basis remains optimal.

Change 2 Changing the objective function coefficient of a basic variable. To see whether the current basis remains optimal, find the new u_i's and v_j's and use these values to price out all nonbasic variables. The current basis remains optimal as long as all nonbasic variables have a nonpositive coefficient in row 0.

Change 3 Increasing both supply s_i and demand d_j by Δ.

$$\text{New } z\text{-value} = \text{old } z\text{-value} + \Delta u_i + \Delta v_j$$

We may find the new values of the decision variables as follows:

1 If x_{ij} is a basic variable in the optimal solution, then increase x_{ij} by Δ.

2 If x_{ij} is a nonbasic variable in the optimal solution, find the loop involving x_{ij} and some of the basic variables. Find an odd cell in the loop that is in row i. Increase the value of this odd cell by Δ and go around the loop, alternately increasing and then decreasing current basic variables in the loop by Δ.

REVIEW PROBLEMS

Group A

1 Televco produces TV picture tubes at three plants. Plant 1 can produce 50 tubes per week; plant 2, 100 tubes per week; and plant 3, 50 tubes per week. Tubes are shipped to three customers. The profit earned per tube depends on the site where the tube was produced and on the customer who purchases the tube (see Table 64). Customer 1 is willing to purchase as many as 80 tubes per week; customer 2, as many as 90; and customer 3, as many as 100. Televco wants to find a shipping and production plan that will maximize profits.

a Formulate a balanced transportation problem that can be used to maximize Televco's profits.

b Use the northwest corner method to find a bfs to the problem.

c Use the transportation simplex to find an optimal solution to the problem.

2 Five workers are available to perform four jobs. The time it takes each worker to perform each job is given in Table 65. The goal is to assign workers to jobs so as to minimize the total time required to perform the four jobs. Use the Hungarian method to solve the problem.

TABLE 64

	To ($)		
From	**Customer 1**	**Customer 2**	**Customer 3**
Plant 1	75	60	69
Plant 2	79	73	68
Plant 3	85	76	70

TABLE 65

	Time (Hours)			
Worker	**Job 1**	**Job 2**	**Job 3**	**Job 4**
1	10	15	10	15
2	12	8	20	16
3	12	9	12	18
4	6	12	15	18
5	16	12	8	12

3 A company must meet the following demands for a product: January, 30 units; February, 30 units; March, 20 units. Demand may be backlogged at a cost of $5/unit/month. All demand must be met by the end of March. Thus, if 1 unit of January demand is met during March, a backlogging cost of 5(2) = $10 is incurred. Monthly production capacity and unit production cost during each month are given in Table 66. A holding cost of $20/unit is assessed on the inventory at the end of each month.

a Formulate a balanced transportation problem that could be used to determine how to minimize the total cost (including backlogging, holding, and production costs) of meeting demand.

b Use Vogel's method to find a basic feasible solution.

c Use the transportation simplex to determine how to meet each month's demand. Make sure to give an interpretation of your optimal solution (for example, 20 units of month 2 demand is met from month 1 production).

4 Appletree Cleaning has five maids. To complete cleaning my house, they must vacuum, clean the kitchen, clean the bathroom, and do general straightening up. The time it takes each maid to do each job is shown in Table 67. Each maid

TABLE 66

Month	**Production Capacity**	**Unit Production Cost**
January	35	$400
February	30	$420
March	35	$410

TABLE 67

	Time (Hours)			
Maid	**Vacuum**	**Clean Kitchen**	**Clean Bathroom**	**Straighten Up**
1	6	5	2	1
2	9	8	7	3
3	8	5	9	4
4	7	7	8	3
5	5	5	6	4

is assigned one job. Use the Hungarian method to determine assignments that minimize the total number of maid-hours needed to clean my house.

5[†] Currently, State University can store 200 files on hard disk, 100 files in computer memory, and 300 files on tape. Users want to store 300 word-processing files, 100 packaged-program files, and 100 data files. Each month a typical word-processing file is accessed eight times; a typical packaged-program file, four times; and a typical data file, two times. When a file is accessed, the time it takes for the file to be retrieved depends on the type of file and on the storage medium (see Table 68).

a If the goal is to minimize the total time per month that users spend accessing their files, formulate a balanced transportation problem that can be used to determine where files should be stored.

b Use the minimum cost method to find a bfs.

c Use the transportation simplex to find an optimal solution.

6 The Gotham City police have just received three calls for police. Five cars are available. The distance (in city blocks) of each car from each call is given in Table 69. Gotham City wants to minimize the total distance cars must travel to respond to the three police calls. Use the Hungarian method to determine which car should respond to which call.

7 There are three school districts in the town of Busville. The number of black and white students in each district are shown in Table 70. The Supreme Court requires the schools in Busville to be racially balanced. Thus, each school must have exactly 300 students, and each school must have the same number of black students. The distances between districts are shown in Table 70.

TABLE 68

	Time (Minutes)		
Storage Medium	**Word Processing**	**Packaged Program**	**Data**
Hard disk	5	4	4
Memory	2	1	1
Tape	10	8	6

TABLE 69

	Distance (Blocks)		
Car	**Call 1**	**Call 2**	**Call 3**
1	10	11	18
2	6	7	7
3	7	8	5
4	5	6	4
5	9	4	7

TABLE 70

	No. of Students		Distance to (Miles)	
District	**Whites**	**Blacks**	**District 2**	**District 3**
1	210	120	3	5
2	210	30	—	4
3	180	150	—	—

Formulate a balanced transportation problem that can be used to determine the minimum total distance that students must be bused while still satisfying the Supreme Court's requirements. Assume that a student who remains in his or her own district will not be bused.

8 Using the northwest corner method to find a bfs, find (via the transportation simplex) an optimal solution to the transportation (minimization) problem shown in Table 71.

9 Solve the following LP:

$$\begin{aligned} \min z &= 2x_1 + 3x_2 + 4x_3 + 3x_4 \\ \text{s.t.} \quad & x_1 + x_2 \le 4 \\ & x_3 + x_4 \le 5 \\ & x_1 + x_3 \ge 3 \\ & x_2 + x_4 \ge 6 \\ \min x_j &\ge 0 \quad (j = 1, 2, 3, 4) \end{aligned}$$

10 Find the optimal solution to the balanced transportation problem in Table 72 (minimization).

11 In Problem 10, suppose we increase s_i to 16 and d_3 to 11. The problem is still balanced, and because 31 units (instead of 30 units) must be shipped, one would think that the total shipping costs would be increased. Show that the total shipping cost has actually decreased by $2, however. This is called the "more for less" paradox. Explain why increasing both the supply and the demand has decreased cost. Using the theory of shadow prices, explain how one could have predicted that increasing s_1 and d_3 by 1 would decrease total cost by $2.

12 Use the northwest corner method, the minimum-cost method, and Vogel's method to find basic feasible solutions to the transportation problem in Table 73.

13 Find the optimal solution to Problem 12.

TABLE 71

12	14	16	60
14	13	19	50
17	15	18	40
40	70	10	

[†] This problem is based on Evans (1984).

TABLE 72

4	2	4	15
12	8	4	15
10	10	10	

TABLE 73

20	11	3	6	5
5	9	10	2	10
18	7	4	1	15
3	3	12	12	

14 Oilco has oil fields in San Diego and Los Angeles. The San Diego field can produce 500,000 barrels per day, and the Los Angeles field can produce 400,000 barrels per day. Oil is sent from the fields to a refinery, either in Dallas or in Houston (assume that each refinery has unlimited capacity). It costs \$700 to refine 100,000 barrels of oil at Dallas and \$900 at Houston. Refined oil is shipped to customers in Chicago and New York. Chicago customers require 400,000 barrels per day of refined oil; New York customers require 300,000. The costs of shipping 100,000 barrels of oil (refined or unrefined) between cities are given in Table 74. Formulate a balanced transportation model of this situation.

15 For the Powerco problem, find the range of values of c_{24} for which the current basis remains optimal.

16 For the Powerco problem, find the range of values of c_{23} for which the current basis remains optimal.

17 A company produces cars in Atlanta, Boston, Chicago, and Los Angeles. The cars are then shipped to warehouses in Memphis, Milwaukee, New York City, Denver, and San Francisco. The number of cars available at each plant is given in Table 75.

Each warehouse needs to have available the number of cars given in Table 76.

The distance (in miles) between the cities is given in Table 77.

- **a** Assuming that the cost (in dollars) of shipping a car equals the distance between two cities, determine an optimal shipping schedule.
- **b** Assuming that the cost (in dollars) of shipping a car equals the square root of the distance between two cities, determine an optimal shipping schedule.

TABLE 74

	To (\$)			
From	Dallas	Houston	N.Y.	Chicago
L.A.	300	110	—	—
San Diego	420	100	—	—
Dallas	—	—	450	550
Houston	—	—	470	530

TABLE 75

Plant	Cars Available
Atlanta	5,000
Boston	6,000
Chicago	4,000
L.A.	3,000

TABLE 76

Warehouse	Cars Required
Memphis	6,000
Milwaukee	4,000
N.Y.	4,000
Denver	2,000
San Francisco	2,000

TABLE 77

	Memphis	Milwaukee	N.Y.	Denver	S.F.
Atlanta	371	761	841	1,398	2,496
Boston	1,296	1,050	206	1,949	3,095
Chicago	530	87	802	996	2,142
L.A.	1,817	2,012	2,786	1,059	379

18 During the next three quarters, Airco faces the following demands for air conditioner compressors: quarter 1—200; quarter 2—300; quarter 3—100. As many as 240 air compressors can be produced during each quarter. Production costs/compressor during each quarter are given in Table 78. The cost of holding an air compressor in inventory is \$100/quarter. Demand may be backlogged (as long as it is met by the end of quarter 3) at a cost of \$60/compressor/quarter. Formulate the tableau for a balanced transportation problem whose solution tells Airco how to minimize the total cost of meeting the demands for quarters 1–3.

19 A company is considering hiring people for four types of jobs. It would like to hire the number of people in Table 79 for each type of job.

Four types of people can be hired by the company. Each type is qualified to perform two types of jobs according to

TABLE 78

Quarter 1	Quarter 2	Quarter 3
$200	$180	$240

TABLE 79

	Job			
	1	2	3	4
Number of people	30	30	40	20

TABLE 80

	Type of Person			
	1	2	3	4
Jobs qualified for	1 and 3	2 and 3	3 and 4	1 and 4

Table 80. A total of 20 Type 1, 30 Type 2, 40 Type 3, and 20 Type 4 people have applied for jobs. Formulate a balanced transportation problem whose solution will tell the company how to maximize the number of employees assigned to suitable jobs. (*Note:* Each person can be assigned to at most one job.)

20 During each of the next two months you can produce as many as 50 units/month of a product at a cost of $12/unit during month 1 and $15/unit during month 2. The customer is willing to buy as many as 60 units/month during each of the next two months. The customer will pay $20/unit during month 1, and $16/unit during month 2. It costs $1/unit to hold a unit in inventory for a month. Formulate a balanced transportation problem whose solution will tell you how to maximize profit.

Group B

21[†] The Carter Caterer Company must have the following number of clean napkins available at the beginning of each of the next four days: day 1—15; day 2—12; day 3—18; day 4—6. After being used, a napkin can be cleaned by one of two methods: fast service or slow service. Fast service costs 10¢ per napkin, and a napkin cleaned via fast service is available for use the day after it is last used. Slow service costs 6¢ per napkin, and these napkins can be reused two days after they are last used. New napkins can be purchased for a cost of 20¢ per napkin. Formulate a balanced transportation problem to minimize the cost of meeting the demand for napkins during the next four days.

22 Braneast Airlines must staff the daily flights between New York and Chicago shown in Table 81. Each of Braneast's crews lives in either New York or Chicago. Each day a crew must fly one New York–Chicago and one Chicago–New York flight with at least 1 hour of downtime between flights. Braneast wants to schedule the crews to minimize the total downtime. Set up an assignment problem that can be used to accomplish this goal. (*Hint:* Let $x_{ij} = 1$ if the crew that flies flight *i* also flies flight *j*, and $x_{ij} = 0$ otherwise. If $x_{ij} = 1$, then a cost c_{ij} is incurred, corresponding to the downtime associated with a crew flying flight *i* and flight *j*.) Of course, some assignments are not possible. Find the flight assignments that minimize the total downtime. How many crews should be based in each city? Assume that at the end of the day, each crew must be in its home city.

TABLE 81

Flight	Leave Chicago	Arrive New York	Flight	Leave New York	Arrive Chicago
1	6 A.M.	10 A.M.	1	7 A.M.	9 A.M.
2	9 A.M.	1 P.M.	2	8 A.M.	10 A.M.
3	12 noon	4 P.M.	3	10 A.M.	12 noon
4	3 P.M.	7 P.M.	4	12 noon	2 P.M.
5	5 P.M.	9 P.M.	5	2 P.M.	4 P.M.
6	7 P.M.	11 P.M.	6	4 P.M.	6 P.M.
7	8 P.M.	12 midnight	7	6 P.M.	8 P.M.

23 A firm producing a single product has three plants and four customers. The three plants will produce 3,000, 5,000, and 5,000 units, respectively, during the next time period. The firm has made a commitment to sell 4,000 units to customer 1, 3,000 units to customer 2, and at least 3,000 units to customer 3. Both customers 3 and 4 also want to buy as many of the remaining units as possible. The profit associated with shipping a unit from plant *i* to customer *j* is given in Table 82. Formulate a balanced transportation problem that can be used to maximize the company's profit.

24 A company can produce as many as 35 units/month. The demands of its primary customers must be met on time each month; if it wishes, the company may also sell units to secondary customers each month. A $1/unit holding cost is assessed against each month's ending inventory. The relevant data are shown in Table 83. Formulate a balanced transportation problem that can be used to maximize profits earned during the next three months.

25 My home has four valuable paintings that are up for sale. Four customers are bidding for the paintings. Customer 1 is willing to buy two paintings, but each other customer is willing to purchase at most one painting. The prices that each customer is willing to pay are given in Table 84. Use

TABLE 82

	To Customer ($)			
From	1	2	3	4
Plant 1	65	63	62	64
Plant 2	68	67	65	62
Plant 3	63	60	59	60

[†] This problem is based on Jacobs (1954).

TABLE 83

Month	Production Cost/Unit ($)	Primary Demand	Available for Secondary Demand	Sales Price/Unit ($)
1	13	20	15	15
2	12	15	20	14
3	13	25	15	16

TABLE 84

	Bid for ($)			
Customer	Painting 1	Painting 2	Painting 3	Painting 4
1	8	11	—	—
2	9	13	12	7
3	9	—	11	—
4	—	—	12	9

the Hungarian method to determine how to maximize the total revenue received from the sale of the paintings.

26 Powerhouse produces capacitors at three locations: Los Angeles, Chicago, and New York. Capacitors are shipped from these locations to public utilities in five regions of the country: northeast (NE), northwest (NW), midwest (MW), southeast (SE), and southwest (SW). The cost of producing and shipping a capacitor from each plant to each region of the country is given in Table 85. Each plant has an annual production capacity of 100,000 capacitors. Each year, each region of the country must receive the following number of capacitors: NE, 55,000; NW, 50,000; MW, 60,000; SE, 60,000; SW, 45,000. Powerhouse feels shipping costs are too high, and the company is therefore considering building one or two more production plants. Possible sites are Atlanta and Houston. The costs of producing a capacitor and shipping it to each region of the country are given in Table 86. It costs $3 million (in current dollars) to build a new plant, and operating each plant incurs a fixed cost (in addition to variable shipping and production costs) of $50,000 per year. A plant at Atlanta or Houston will have the capacity to produce 100,000 capacitors per year.

Assume that future demand patterns and production costs will remain unchanged. If costs are discounted at a rate of $11\frac{1}{9}\%$ per year, how can Powerhouse minimize the present value of all costs associated with meeting current and future demands?

TABLE 85

	To ($)				
From	NE	NW	MW	SE	SW
L.A.	27.86	4.00	20.54	21.52	13.87
Chicago	8.02	20.54	2.00	6.74	10.67
N.Y.	2.00	27.86	8.02	8.41	15.20

TABLE 86

	To ($)				
From	NE	NW	MW	SE	SW
Atlanta	8.41	21.52	6.74	3.00	7.89
Houston	15.20	13.87	10.67	7.89	3.00

27[†] During the month of July, Pittsburgh resident B. Fly must make four round-trip flights between Pittsburgh and Chicago. The dates of the trips are as shown in Table 87. B. Fly must purchase four round-trip tickets. Without a discounted fare, a round-trip ticket between Pittsburgh and Chicago costs $500. If Fly's stay in a city includes a weekend, then he gets a 20% discount on the round-trip fare. If his stay in a city is at least 21 days, then he receives a 35% discount; and if his stay is more than 10 days, then he receives a 30% discount. Of course, only one discount can be applied toward the purchase of any ticket. Formulate and solve an assignment problem that minimizes the total cost of purchasing the four round-trip tickets. (*Hint:* Let $x_{ij} = 1$ if a round-trip ticket is purchased for use on the *i*th flight out of Pittsburgh and the *j*th flight out of Chicago. Also think about where Fly should buy a ticket if, for example, $x_{21} = 1$.)

28 Three professors must be assigned to teach six sections of finance. Each professor must teach two sections of finance, and each has ranked the six time periods during which finance is taught, as shown in Table 88. A ranking of 10 means that the professor wants to teach that time, and a ranking of 1 means that he or she does not want to teach at that time. Determine an assignment of professors to sections that will maximize the total satisfaction of the professors.

29[‡] Three fires have just broken out in New York. Fires 1 and 2 each require two fire engines, and fire 3 requires three fire engines. The "cost" of responding to each fire depends on the time at which the fire engines arrive. Let t_{ij} be the time (in minutes) when the *j*th engine arrives at fire *i*. Then the cost of responding to each fire is as follows:

$$\begin{aligned} &\text{Fire 1:} \quad 6t_{11} + 4t_{12} \\ &\text{Fire 2:} \quad 7t_{21} + 3t_{22} \\ &\text{Fire 3:} \quad 9t_{31} + 8t_{32} + 5t_{33} \end{aligned}$$

Three fire companies can respond to the three fires. Company 1 has three engines available, and companies 2

TABLE 87

Leave Pittsburgh	Leave Chicago
Monday, July 1	Friday, July 5
Tuesday, July 9	Thursday, July 11
Monday, July 15	Friday, July 19
Wednesday, July 24	Thursday, July 25

[†] Based on Hansen and Wendell (1982).

[‡] Based on Denardo, Rothblum, and Swersey (1988).

TABLE 88

Professor	9 A.M.	10 A.M.	11 A.M.	1 P.M.	2 P.M.	3 P.M.
1	8	7	6	5	7	6
2	9	9	8	8	4	4
3	7	6	9	6	9	9

TABLE 89

Company	Fire 1	Fire 2	Fire 3
1	6	7	9
2	5	8	11
3	6	9	10

and 3 each have two engines available. The time (in minutes) it takes an engine to travel from each company to each fire is shown in Table 89.

a Formulate and solve a transportation problem that can be used to minimize the cost associated with assigning the fire engines. (*Hint:* Seven demand points will be needed.)

b Would the formulation in part (a) still be valid if the cost of fire 1 were $4t_{11} + 6t_{12}$?

REFERENCES

The following six texts discuss transportation, assignment, and transshipment problems:

Bazaraa, M., and J. Jarvis. *Linear Programming and Network Flows.* New York: Wiley, 1990.

Bradley, S., A. Hax, and T. Magnanti. *Applied Mathematical Programming.* Reading, Mass.: Addison-Wesley, 1977.

Dantzig, G. *Linear Programming and Extensions.* Princeton, N.J.: Princeton University Press, 1963.

Gass, S. *Linear Programming: Methods and Applications,* 5th ed. New York: McGraw-Hill, 1985.

Murty, K. *Linear Programming.* New York: Wiley, 1983.

Wu, N., and R. Coppins. *Linear Programming and Extensions.* New York: McGraw-Hill, 1981.

Aarvik, O., and P. Randolph. "The Application of Linear Programming to the Determination of Transmission Line Fees in an Electrical Power Network," *Interfaces* 6(1975):17–31.

Denardo, E., U. Rothblum, and A. Swersey. "Transportation Problem in Which Costs Depend on Order of Arrival," *Management Science* 34(1988):774–784.

Evans, J. "The Factored Transportation Problem," *Management Science* 30(1984):1021–1024.

Gillett, B. *Introduction to Operations Research: A Computer-Oriented Algorithmic Approach.* New York: McGraw-Hill, 1976.

Glassey, R., and V. Gupta. "A Linear Programming Analysis of Paper Recycling," *Management Science* 21(1974): 392–408.

Glover, F., et al. "A Computational Study on Starting Procedures, Basis Change Criteria and Solution Algorithms for Transportation Problems," *Management Science* 20(1974):793–813. This article discusses the computational efficiency of various methods used to find basic feasible solutions for transportation problems.

Hansen, P., and R. Wendell. "A Note on Airline Commuting," *Interfaces* 11(no. 12, 1982):85–87.

Jackson, B. "Using LP for Crude Oil Sales at Elk Hills: A Case Study," *Interfaces* 10(1980):65–70.

Jacobs, W. "The Caterer Problem," *Naval Logistics Research Quarterly* 1(1954):154–165.

Machol, R. "An Application of the Assignment Problem," *Operations Research* 18(1970):745–746.

Srinivasan, P. "A Transshipment Model for Cash Management Decisions," *Management Science* 20(1974): 1350–1363.

Wagner, H., and D. Rubin. "Shadow Prices: Tips and Traps for Managers and Instructors," *Interfaces* 20(no. 4, 1990):150–157.