CHAPTER 5

EIGEN VALUE, EIGEN VECTOR & CAYLEY HAMILTON THEOREM

In this chapter we will discuss a very important concept of linear algebra i.e **Eigen Value and Eigen Vector** which has vast application in many discipline both in engineering and science. The Control theory, vibration analysis, electric circuits, advanced dynamics and quantum mechanics, coding theory, cryptography, image processing are just a few of the application areas of it.

5.1 Eigen Value and Eigen Vector:

If A is a $n \times n$ matrix and λ an indeterminate. The matrix [A- λI] is called the **characteristic matrix** of A where I is the unit matrix of order $n \times n$.

The determinant of the matrix [A- λI], i.e. $|A - \lambda I|$ is called the **characteristic polynomial** of A which is an ordinary polynomial in λ of degree n,.

The equation $|A - \lambda I| = 0$ is called the **characteristic equation** of A and the roots of this equation are called the characteristic roots or characteristic values or Eigen values or latent roots of the matrix A.

Definition Let A be a $n \times n$ matrix. A scalar λ is called an **eigenvalue** of matrix A if there exists a vector $X \neq 0$ such that

$$AX = \lambda X$$

Definition If λ is a characteristic root/eigenvalue of the matrix A, then $|A - \lambda I| = 0$ or the matrix A – λI is singular. Hence there exists a non-zero vector X such that

$$(A- \lambda I)\, X = O \qquad or \qquad AX = \lambda X.$$

Such a vector X, is said to be an **eigenvector** associated with the eigenvalue λ.

Important facts

i. The set of the eigenvalues of A is called the spectrum of A.

ii. The sum of the elements on the principal diagonal of a matrix is called the trace of the matrix.

iii. The eigenvectors corresponding to distinct eigenvalues are always linearly independent.

Process to find the eigenvalues and eigenvectors:

If A is a $n \times n$ matrix and λ an indeterminate then $|A - \lambda I_n|$ is a polynomial in λ, so you get the eigenvalues of A by finding the roots of $|A - \lambda I_n| = 0$ .Suppose that λ is an eigenvalue for A. Then there exists a non-zero vector $X = \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ such that $AX = \lambda X$ or, equivalently, such that $(A - \lambda I_n)X = 0$, where I_n is the $n \times n$ identity matrix.

Say λ_0 is such an eigenvalue. In order to find the eigenvectors associated with λ_0, you have to solve the system $AX = \lambda_0 X$ for $x_1, ..., x_n$.

SOLVED EXAMPLES

Example 1 Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ find the eigenvalues and eigen vectors of A,

Solution To find the eigenvalues construct the matrix $A - \lambda I_2$, find its determinant $|A - \lambda I_2|$ and solve the equation $|A - \lambda I_2| = 0$:

$$A - \lambda I_2 = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix}.$$

Now, $|A - \lambda I_2| = \lambda^2 - 6\lambda + 5$,

so that the eigenvalues of A are the roots of $|A - \lambda I_2| = 0$:

$\lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0$, i.e. A has eigenvalues 5 & 1

Now find the eigen vector for $\lambda_0 = 1$.

We have to solve the system $AX = \lambda_0 X$. Here $\lambda_0 = 1$, so that

$$AX = \lambda_0 X \quad \Leftrightarrow \quad AX = X,$$

which yields
$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

A vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is therefore an eigenvector associated with the eigenvalue $\lambda_0 = 1$ if and only if its coordinates satisfy

$$\begin{matrix} 1x_1 & +3x_2 = 0 \\ 1x_1 & +3x_2 = 0 \end{matrix},$$

i.e. if and only if $x_1 = -3x_2$.

Hence the eigenvectors associated with the eigenvalue $\lambda_0 = 1$ is $X = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$,

Now find the eigen vector for $\lambda_1 = 5$.

We have to solve the system $AX = \lambda_1 X$. Here $\lambda_1 = 5$, so that

which yields
$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is therefore an eigenvector associated with the eigenvalue $\lambda_1 = 5$ if and only if its coordinates satisfy

$$\begin{matrix} -3x_1 & +3x_2 = 0 \\ 1x_1 & -1x_2 = 0 \end{matrix},$$

i.e. if and only if $x_1 = x_2$.

Hence the eigenvectors associated with the eigenvalue $\lambda_1 = 1$ is $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

Example 2 Let $A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$ find the eigenvalues and eigenvectors of A,

Solution To find the eigenvalues construct the matrix $A - \lambda I_2$, find its determinant $|A - \lambda I_2|$ and solve the equation $|A - \lambda I_2| = 0$:

$$A - \lambda I_2 = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix}.$$

Now, $|A - \lambda I_2| = (-3-\lambda)(1-\lambda) - (2)(-2) = \lambda^2 + 2\lambda + 1$,

so that the eigenvalues of A are the roots of $|A - \lambda I_2| = 0$:

$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$, i.e. A has a repeated eigenvalue -1 & -1

Now find the eigen vector for $\lambda_0 = -1$.

We have to solve the system $AX = \lambda_0 X$. Here $\lambda_0 = -1$, so that

$$AX = \lambda_0 X \quad \Leftrightarrow \quad AX = -X,$$

which yields
$$\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or, equivalently
$$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is therefore an eigenvector associated with the eigenvalue $\lambda_0 = -1$ if and only if its coordinates satisfy

$$\begin{array}{rr} -2x_1 & +2x_2 = 0 \\ -2x_1 & +2x_2 = 0 \end{array},$$

i.e. if and only if $x_1 = x_2$.

Hence the eigenvectors associated with the eigenvalue $\lambda_0 = -1$ are of the form $X = \alpha\begin{pmatrix}1\\1\end{pmatrix}$,

where α is a real number. So we can change the value of α to get infinitely many linearly dependent eigen vectors. But there is only one independent eigen vector exist for the matrix A corresponding the repeated eigen value $\lambda_0 = -1$

Example 3 Find eigen values and eigen vectors for the matrix A

$$A = \begin{pmatrix}2 & 3 & 0\\1 & 4 & 0\\0 & 0 & 5\end{pmatrix}.$$

Solution To find the eigenvalues construct the matrix $A - \lambda I_3$, find its determinant $|A - \lambda I_3|$ and solve the equation $|A - \lambda I_3| = 0$:

Hence $A - \lambda I_3 = \begin{pmatrix}2-\lambda & 3 & 0\\1 & 4-\lambda & 0\\0 & 0 & 5-\lambda\end{pmatrix}$

so that the eigenvalues of A are the roots of $|A - \lambda I_3| = 0$:

hence $|A - \lambda I_3| = (5-\lambda)(5-\lambda)(1-\lambda)$ =0.

A therefore has three eigenvalues: $\lambda_0 = 1$ $\lambda_1 = 5{,}5$

To get the eigenvectors associated with $\lambda_0 = 1$, we solve $(A - 1I_3)X = 0$:

$$(A-4I_3)X = \begin{pmatrix}1 & 3 & 0\\1 & 3 & 0\\0 & 0 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow \begin{cases}x_1 & +3x_2 & +0x_3 = 0\\x_1 & +3x_2 & +0x_3 = 0\\0x_1 & +0x_2 & 4x_3 = 0\end{cases} \Leftrightarrow \begin{cases}x_1 = -3x_2\\x_3 = 0\end{cases}.\text{Hence}$$

the eigenvectors associated with the eigenvalue $\lambda_0 = 1$ are of the form $\alpha\begin{pmatrix}-3\\1\\0\end{pmatrix}$.

To get the eigenvectors associated with $\lambda_1 = 5$, we solve $(A - 5I_3)X = 0$:

$$(A-4I_3)X = \begin{pmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3x_1 & +3x_2 & +0x_3 = 0 \\ x_1 & -1x_2 & +0x_3 = 0 \\ 0x_1 & +0x_2 & +0x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 \\ x_3 = const. \end{cases}.$$

Hence the eigenvectors associated with the eigenvalues $\lambda_1 = 5{,}5$ are of the form $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} and \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Hence there exist two linearly independent eigenvectors associated with the repeated eigenvalue $\lambda_0 = 5$

Example 4 Find eigen values and eigen vectors for the matrix A

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

Solution To find the eigenvalues construct the matrix $A - \lambda I_3$, find its determinant $|A - \lambda I_3|$ and solve the equation $|A - \lambda I_3| = 0$:

Hence $A - \lambda I_3 = \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix}$

And $|A - \lambda I_3| = (-3-\lambda)\begin{vmatrix} 5-\lambda & -1 \\ 6 & -2-\lambda \end{vmatrix} - \begin{vmatrix} -7 & -1 \\ -6 & -2-\lambda \end{vmatrix} - \begin{vmatrix} -7 & 5-\lambda \\ -6 & 6 \end{vmatrix} = -\lambda^3 + 12\lambda + 16$.

so that the eigenvalues of A are the roots of $|A - \lambda I_3| = 0$:

$$\lambda^3 - 12\lambda - 16 = 0$$

$\lambda = -2$, you get $\qquad (-2)^3 - 12(-2) - 16 = 0$,

so that $\lambda = -2$ is a root of $\qquad |A - \lambda I_3| = 0$

Hence $(\lambda+2)$ is a factor of $|A-\lambda I_3|$ and $|A-\lambda I_3|$ can be written in the form

$$|A-\lambda I_3| = (\lambda+2)(-\lambda^2+2\lambda+8) = -(\lambda+2)^2(\lambda-4) = 0.$$

A therefore has two more eigenvalues: $\lambda_0 = -2$ and $\lambda_1 = 4$.

A therefore has three eigenvalues: $\lambda_0 = -2$, -2 and $\lambda_1 = 4$.

To get the eigenvectors associated with $\lambda_0 = -2$, we solve $(A+2I_3)X = 0$:

$$(A+2I_3)X = \begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -x_1 \quad +x_2 \quad -x_3 = 0 \\ -7x_1 \quad +7x_2 \quad -x_3 = 0 \\ -6x_1 \quad +6x_2 \quad = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 \\ x_3 = 0 \end{cases}.$$

Hence the eigenvectors associated with the eigenvalue $\lambda_0 = -2$ are of the form $\alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

To get the eigenvectors associated with $\lambda_1 = 4$, we solve $(A-4I_3)X = 0$:

$$(A-4I_3)X = \begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -7x_1 \quad +x_2 \quad -x_3 = 0 \\ -7x_1 \quad +x_2 \quad -x_3 = 0 \\ -6x_1 \quad +6x_2 \quad -6x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = x_3 \end{cases}.$$

Hence the eigenvectors associated with the eigenvalue $\lambda_1 = 4$ are of the form $\alpha\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Important Results of eigenvalues and eigenvectors

1. If A is a $n \times n$ triangular matrix – upper triangular, lower triangular or diagonal, the eigenvalues of A are the diagonal entries of A.

2. $\lambda = 0$ is an eigenvalue of A if A is a singular (noninvertible) matrix.

3. A and A^{T} have the same eigenvalues.

4. Eigenvectors of a symmetric matrix are orthogonal, but only for distinct eigenvalues.

5. $|\det(A)|$ is the product of the absolute values of the eigenvalues of A.

Theorem 1 The eigenvalues of a real symmetric matrix are real.

Proof Suppose that x is an eigenvector of A corresponding to the eigenvalue λ, we have

$$Ax = \lambda x$$

Take the complex conjugate of both sides

$$Ax = \lambda x \Rightarrow \overline{\mathbf{A}}\overline{\mathbf{x}} = \lambda^* \overline{\mathbf{x}} \Rightarrow A\,\overline{\mathbf{x}} = \lambda^* \overline{\mathbf{x}} \text{ since A is a real matrix}$$

$$(Ax)^T \overline{\mathbf{x}} = (\lambda x)^T \overline{\mathbf{x}} \Rightarrow x^T(A^T \overline{\mathbf{x}}) = \lambda x^T \overline{\mathbf{x}}$$

$$x^T(A\,\overline{\mathbf{x}}) = \lambda x^T \overline{\mathbf{x}} \Rightarrow x^T(\lambda^* \overline{\mathbf{x}}) = \lambda x^T \overline{\mathbf{x}}$$

$$\lambda^* x^T \overline{\mathbf{x}} = \lambda x^T \overline{\mathbf{x}} \Rightarrow \lambda = \lambda^* \text{ so that the eigenvalues are real.}$$

Example 5 find the eigenvalues of

$$A = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 9 & 5 & 7.5 & 0 \\ 2 & 6 & 0 & -7.2 \end{bmatrix}$$

Solution

Since the matrix A is a lower triangular matrix, the eigenvalues of A are the diagonal elements of A.

The eigenvalues are $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 7.5, \lambda_4 = -7.2$

Example 6 One of the eigenvalues of $A = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 5 & 9 \\ 2 & 1 & -7 \end{bmatrix}$ is zero. Is A invertible?

Solution Since one of the eigen values is zero

hence $\lambda = 0$ is an eigenvalue of A,

that implies A is singular and is not invertible.

Example 7 Given the eigenvalues of $A = \begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix}$

Are $\lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$.

Example 8 What are the eigenvalues of $B = \begin{bmatrix} 2 & 3.5 & 8 \\ -3.5 & 5 & 1 \\ 6 & 2 & 8.5 \end{bmatrix}$?

Solution Since $B = A^T$, the eigenvalues of A and B are the same. Hence eigenvalues of B also are $\lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$

Example 9 Given the eigenvalues of $A = \begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix}$ are

$\lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$, Calculate the magnitude of the determinant of the matrix.

Solution Since $|\det A| = |\lambda_1||\lambda_2||\lambda_3|$

$$= |-1.547||12.33||4.711| \quad = 89.88$$

5.2 The Cayley-Hamilton Theorem:

Every square matrix satisfies its own characteristic equation.

i.e., Let A **is a** $n \times n$ **, then its characteristic equation is** $|A - \lambda I| = 0$

$$|A - \lambda I| = (-1)^n [\lambda_n + a_1 \lambda_{n-1} + a_2 \lambda_{n-2} + \ldots + a_n] = 0,$$

Then the matrix equation

$$X_n + a_1X_{n-1} + a_2X_{n-2} + a_3X_{n-3} + \ldots\ldots + a_nI = O.$$

Is satisfied by X = A

i.e., $A_n + a_1A_{n-1} + a_2A_{n-2} + \dots + a_nI = O.$

Find A^{-1} using Cayley Hamilton theorem

Let A is a $n \times n$ non singular matrix, then its characteristic equation is

$$|A - \lambda I| = f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

Since **A** is a non singular matrix, $\lambda_i \neq 0$; that is, every eigenvalue is nonzero, and

$$(-1)^n a_0 = |A| \neq 0.$$

Therefore as per Cayley Hamilton Theorem, A satisfies its characteristic equation

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \dots + a_1\mathbf{A} + a_0\mathbf{I} = 0$$

$$\Rightarrow \mathbf{I} = -\frac{1}{a_0}(\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \dots + a_1\mathbf{A})$$

Multiply $\mathbf{A}^{-1}$ both sides, we get

$$\mathbf{A}^{-1} = -\frac{1}{a_0}(\mathbf{A}^{n-1} + a_{n-1}\mathbf{A}^{n-2} + \dots + a_1\mathbf{I})$$

SOLVED EXAMPLES

Example 1 Verify the Cayley-Hamilton theorem for the matrix A= $\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ and hence find A^{-1}.

Solution we have

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

The characteristic equation of the matrix is

$$|A - \lambda I| = 0$$

$$\left|\begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right| = 0$$

$$\begin{vmatrix}2-\lambda & 2 & 1\\ 0 & 1-\lambda & -1\\ 3 & -1 & 1-\lambda\end{vmatrix} = 0$$

$$\lambda^3 - 4\lambda^2 + \lambda + 9 = 0$$

As per Cayley Hamilton Theorem, A satisfies its characteristic equation

Hence $$A^3 - 4A^2 + A + 9I = 0$$

Now calculate A^3, A^2 and put in above equation

$$A^2 = A \times A = \begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix} \times \begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix} = \begin{bmatrix}7 & 5 & 1\\ -3 & 2 & -2\\ 9 & 4 & 5\end{bmatrix}$$

$$A^3 = A \times A \times A = \begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix} \times \begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix} \times \begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}7 & 5 & 1\\ -3 & 2 & -2\\ 9 & 4 & 5\end{bmatrix} \times \begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}17 & 18 & 3\\ -12 & -2 & -7\\ 33 & 17 & 10\end{bmatrix}$$

Now to verify

$A^3 - 4A^2 + A + 9I = 0$

$$= \begin{bmatrix}17 & 18 & 3\\ -12 & -2 & -7\\ 33 & 17 & 10\end{bmatrix} - 4\begin{bmatrix}7 & 5 & 1\\ -3 & 2 & -2\\ 9 & 4 & 5\end{bmatrix} + \begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix} + 9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}17-28+2+9 & 18-20+2+0 & 3-4+1+0\\ -12-12+0+0 & -2-8+1+9 & -7+8-1+0\\ 33-36+3+0 & 17-16-1+0 & 10-20+1+19\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Hence Cayley Hamilton theorem is verified.

Now to find A^{-1} multiply the e.q.1 by A^{-1}, we have

$$A^{-1} \times (A^3 - 4A^2 + A + 9I\ ) = 0 \times A^{-1}$$

$$(A^2 - 4A + I + 9A^{-1}\ ) = 0$$

$$A^{-1} = -\frac{1}{9}(A^2 - 4A\quad + I)$$

$$= \frac{-1}{9}\left\{\begin{bmatrix} 7 & 5 & 1 \\ -3 & 2 & -2 \\ 9 & 4 & 5 \end{bmatrix} - 4\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$Hence\ A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 1 & -2 \\ 3 & -8 & -2 \end{bmatrix}$$

Example 2 Verify the Cayley-Hamilton theorem for the matrix A= $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$ and hence find A^{-1}.

Solution we have

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$$

The **characteristic equation of the matrix is**

$$|A - \lambda I| = 0$$

$$\left|\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 9\lambda^2 - 9\lambda + 81 = 0$$

As per Cayley Hamilton Theorem, A satisfies its characteristic equation

Hence $$A^3 - 9A^2 - 9A + 81I = 0$$

Now calculate A^3, A^2 and put in above equation

$$A^2 = A \times A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 17 & -16 & -16 \\ -16 & 41 & 32 \\ -16 & 32 & 41 \end{bmatrix}$$

$$A^3 = A \times A \times A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 81 & -144 & -180 \\ -144 & 333 & 324 \\ -180 & 324 & 315 \end{bmatrix}$$

Now to verify

$A^3 - 9A^2 - 9A + 81I$

$$= \begin{bmatrix} 81 & -144 & -180 \\ -144 & 333 & 324 \\ -180 & 324 & 315 \end{bmatrix} - 9\begin{bmatrix} 17 & -16 & -16 \\ -16 & 41 & 32 \\ -16 & 32 & 41 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} + 81\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley Hamilton theorem is verified.

Now to find A^{-1} multiply the eq1 by A^{-1}, we have

$$\mathrm{A}^{-1} \times (A^3 - 9A^2 - 9A + 81I\ ) = 0 \times \mathrm{A}^{-1}$$

$$(A^2 - 9A - 9I + 81\mathrm{A}^{-1}\ ) = 0$$

$$A^{-1} = -\frac{1}{81}(A^2 - 9A \quad - 9I)$$

$$= \frac{-1}{81}\left\{\begin{bmatrix} 17 & -16 & -16 \\ -16 & 41 & 32 \\ -16 & 32 & 41 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$Hence\ A^{-1} = \frac{1}{81}\begin{bmatrix} 1 & 16 & -20 \\ 16 & 13 & 4 \\ -20 & 4 & -5 \end{bmatrix}$$

Example 3 Using the Cayley-Hamilton theorem find A^{-1}.for the matrix $A=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

Solution we have

$$A=\begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}$$

The characteristic equation of the matrix is

$$|A-\lambda I|=0$$

$$\left|\begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right|=0$$

$$\begin{vmatrix}1-\lambda & 2 & 1\\ 0 & 3-\lambda & 2\\ 1 & 0 & 1-\lambda\end{vmatrix}=0$$

$$\lambda^3-5\lambda^2+6\lambda-4=0$$

As per Cayley Hamilton Theorem, A satisfies its characteristic equation

Hence $$A^3-5A^2+6A-4I=0$$

Now calculate A^3, A^2 and put in above equation

$$A^2=A\times A=\begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 8 & 6\\ 2 & 9 & 8\\ 2 & 2 & 2\end{bmatrix}$$

Now to find A^{-1} multiply the e.q.1 by A^{-1}, we have

$$A^{-1}\times(A^3-5A^2+6A-4I)=0\times A^{-1}$$

$$(A^2-5A+6I-4A^{-1})=0$$

$$A^{-1}=\frac{1}{4}(A^2-5A+6I)$$

$$=\frac{1}{4}\left\{\begin{bmatrix}2 & 8 & 6\\ 2 & 9 & 8\\ 2 & 2 & 2\end{bmatrix}-5\begin{bmatrix}2 & 2 & 1\\ 0 & 1 & -1\\ 3 & -1 & 1\end{bmatrix}+6\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}$$

$$Hence\ A^{-1} = \frac{1}{9}\begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -2 \\ -3 & 2 & 3 \end{bmatrix}$$

EXERCISES

1. Find the eigenvalues and the eigenvectors of the matrix A .where $A=\begin{pmatrix}0 & 2\\ 3 & 5\end{pmatrix}$

   Ans: -2, 3, $\begin{pmatrix}-1\\ 1\end{pmatrix}, \begin{pmatrix}2\\ 5\end{pmatrix},$

2. Find the eigenvalues and the eigenvectors of the matrix A .where $A=\begin{pmatrix}0 & 5\\ 0 & 1\end{pmatrix}$

   Ans: 0, 1, $\begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}5\\ 1\end{pmatrix},$

3. Find the eigenvalues and the eigenvectors of the matrix A .where

   $A=\begin{pmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{pmatrix}$ Ans: 4, -2,-2 $\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}, \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$

4. Find the eigen values and eigen vectors of the matrix A= $\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$.

   Ans: **4, -2,-2** $\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}, \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$

5. Define the eigen values, eigen vectors and the characteristic equation of a square matrix. Find the characteristic equation/polynomial, eigen values and eigen vectors of the matrix $\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{-2}\\ \mathbf{1} & \mathbf{2} & \mathbf{1}\\ \mathbf{1} & \mathbf{0} & \mathbf{3}\end{bmatrix}$.

   Ans: 1, 2,2 $\begin{pmatrix}-2\\ 1\\ 1\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$

6. Find the characteristic equation of the matrix $\begin{bmatrix}1 & 2 & 2\\ 0 & 2 & 1\\ -1 & 2 & 2\end{bmatrix}$. Also find the eigen values and eigen vectors of this matrix.

Ans: 1, 2,2 $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},$

7. Find the eigen values and eigen vectors of the matrix A= $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Ans: 0, 3,15 $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

8. Verify the Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$. Also, find its inverse using this theorem.

Ans:

9. Verify Cayley-Hamilton theorem for the matrix A= $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and hence find A^{-1}.

10. Verify the Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Also, find its inverse using this theorem.

11. Verify Cayley-Hamilton theorem for A and hence find A^{-1} when A= $\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$