CHAPTER – 3

RANK OF A MATRIX.

3.1 Sub Matrix A matrix obtained from a matrix A by omitting some of its rows or columns
(or both) is called a sub matrix of A

For Example the matrices

$$A_1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 & 9 \\ 3 & 4 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 9 & 5 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 2 & 1 & 9 \\ 3 & 4 & 1 \\ 5 & 7 & 2 \\ 4 & 1 & 7 \end{bmatrix}$$ etc. are the sub matrices of a matrix $A = \begin{bmatrix} 2 & 1 & 9 & 5 \\ 3 & 4 & 1 & 0 \\ 5 & 7 & 2 & 1 \\ 4 & 1 & 7 & 7 \end{bmatrix}$ in which A_1 and A_3 are square sur matrices of of order 2 and 3 respectively of A .

As in care of set, every matrix is considers as in sub matrix of itself

3.2 Minor of A Matrix The determinant of a square sub matrix of order r is called a minor of order r of A. A minor of order r is said to be r-rowed minor.

For Example-Consider a matrix $A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 2 & 0 & 4 \\ 3 & 3 & 2 & 6 \end{bmatrix}$

On omitting forth column , we get a square sub matrix $A_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix}$ and its determinant

i.e.$|A_1| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 3 & 3 & 2 \end{vmatrix} = 2$ is a minor of order 3

On omitting third columns we get square matrix

$A_2 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}$ and $|A_2| = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 2 & 4 \\ 3 & 3 & 6 \end{vmatrix}$ is also a minor of order 3

On omitting third row and third & forth columns , we get the square sub matrix $A_3 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $|A_3|$ is minor of order 2 .

On omitting second row and second & third columns we get square matrix $A_4 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $|A_4|$ is again a minor of order 2 since $|A_1| = 2 \neq 0$ &$|A_3| = 4 \neq 0$, then these are called non zero (or non vanishing) minors whereas $|A_2| = 0$ &$|A_4| = 0,$ so these are called zero (or vanishing) minors of A

Remarks (**a**) from this Example it is also clear that there may be more than one minors of the same order obtained by deleting different rows and\or columns

(**b**) The order of the largest order minor of a matrix of order $m \times n =$ min.(m, n)

For Example the order of the largest order minor of a matrix $\begin{bmatrix} 5 & 2 & 3 & 1 \\ 4 & 6 & 2 & 8 \\ 9 & 5 & 6 & 2 \end{bmatrix}_{3\times4}$ is 3

and (**c**) the order of the largest order minor of a n-square matrix = n

e.g. the order of the largest order minor of $\begin{bmatrix} 2 & 3 & 1 & 5 \\ 1 & 0 & 3 & 1 \\ 2 & 6 & 4 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix}$ is 4 .

3.3 RANK OF A MATRIX

The rank of a matrix A is r (a positive integer) if at least one minor of order r of A is non zero whereas it's all minors of order $(r + 1)$ are zero. The rank of a matrix A denoted by $r(A)$ or $\rho(A)$.

Important Facts

- **i.** For a non zero matrix A, $\rho(A) \geq 1$
- **ii.** The rank of a null matrix of any order is zero
- **iii.** The rank of an $m \times n$ matrix A is less than or equal to the min. {m,n} i,e.$\rho(A) \leq$ min. $\{m, n\}$
- **iv.** The rank of a non- singular square matrix is always equal to its order.
- **v.** $\rho(I_n) = n,$ where In is a unit matrix of order
- **vi.** $\rho(A') = \rho(A) = \rho(A^{\theta})$
- **vii.** The rank of the product of two matrices cannot exceed the rank of either matrix i.e. $\rho(AB) \leq \rho(A)$ or $\rho(B)$
- **viii.** The rank of a matrix remains unchanged by applying elementary transformations i.e. the equivalent matrices have the same rank
- **ix.** Let A is a non zero column matrix and B is a non zero row matrix then $\rho(AB) = 1$.
- **x.** If A and B be two equivalent matrices, then rank A = rank B.

xi. If A and B are equivalent matrices, then there exist non-singular matrices P and Q such that B = PAQ.

xii. If two matrices A and B have the same size and the same rank, they are equivalent.

xiii. Two m x n matrices are equivalent if and only if they have the same rank.

3.3.1 Nullity of a matrix- If A is a square matrix of order n then $n-\rho(A)$ is called the nullity of the matrix A and is denoted by $N(A)$. Thus the nullity of a non-singular square matrix of order n is zero.

3.4 METHODS TO FIND RANK OF A MATRIX

The methods, which are often used of find rank of the matrices, can be described as

1. By using definition with or without means of the elementary transformations

2. By reducing into normal form

3. By reducing into echelon form.

1. By Using Definitions –

From definition of rank, the rank of a matrix A is the order of the largest order non zero minor of A so that to find the rank of a matrix A, we have to identify the order of the largest non- zero minor .But in the case of matrices of large order, this process, involves a lot of computations, So it is tedious.

To reduce computations, we transform maximum possible entries of A to zero by applying elementary transformations and then use definition.

SOLVED EXAMPLES

Example 1 Find the rank of matrix $A=\begin{bmatrix}5 & -1 & 0\\ 3 & 1 & 2\\ 2 & 3 & 6\end{bmatrix}$

Solution- Since A is a square matrix then we have

$$|A|=\begin{vmatrix}5 & -1 & 0\\ 3 & 1 & 2\\ 2 & 3 & 6\end{vmatrix}=5(6-6)+1(18-4)+0.(9-2)$$

$$= 0 + 14 + 0$$

$$= 14 \neq 0$$

i.e.$|A| \neq 0$ which is the largest non zero minor of A therefore$(A) = 3$

Example 2 Find the rank of the following matrices

i. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

ii. $B = \begin{bmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{bmatrix}$

iii. $C = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$

Solution (i) we have $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix}$

$$= 1(24 - 25) - 2(18 - 20) + 3(15 - 16)$$

$= -1 + 4 - 3 = 0$ $|A| = 0$ i.e, minor

$|A| = 0$ i.e, minor of order 3 is zero

$\Rightarrow e(A) < 3$

Further the minor of order 2 obtained by deleting third row & third column

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$$

Hence $\rho(A) = 2$

(ii) We have $|B| = \begin{vmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{vmatrix} = 0(0 + i)^2 - i(0 - i)^2 - i(i^2 - 0)$

$= 0 + i^3 - i^3 = 0$

$\therefore \rho(B) < 3$

And the minor of 2 obtained by deleting the first row & third column

$$\begin{vmatrix} -i & 0 \\ i & -i \end{vmatrix} = i^2 - 0 = -1 \neq 0$$

Hence $\rho(B) = 2$

(iii) It is obvious that $|C| = 0$ and also all minors of order 2 are zero but each element of C is not zero, therefore $\rho(C) = 1$

Example 3 Find the rank of matrix : A = $\begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1 \ , R_3 \to R_3 - R_1$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 7 & 13 & -9 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the number of non-zero rows is 2 , therefore Rank (A) = 2 .

Example 4 Find the rank of matrix : A = $\begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$

Solution: $$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1 \ , R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 7 & 13 & -9 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the number of non-zero rows is 2 , therefore Rank (A) = 2 .

2. **By Reducing Into Normal Form**

Any non-zero matrix A of rank r can be reduced by a sequence of elementary transformations to the Form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $I_r$is a unit matrix of order r. This form is called a normal form of A.

Other normal forms are $I_r, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, [I_r, 0]$.

Theorem 1 Let A be an m × n matrix of rank r. Then there exist non-singular matrices P and Q of orders m and n respectively such that PAQ =$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

NOTE-

(i) Each elementary row transformation of A is equivalent to pre multiplying A by the corresponding elementary matrix.

(ii) Each elementary column transformation is equivalent to post multiplying A by the corresponding elementary matrix.

So, there exist elementary matrices $P_1, P_2, \dots, P_k$ and $Q_1, Q_2 \dots Q_t$such that

$P_1, P_2, \dots, P_k$ A $Q_1, Q_2 \dots Q_t = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where $P = P_1, P_2, \dots, P_k \& Q = Q_1, Q_2 \dots Q_t,$

Working Rule to Find Normal Form and Non Singular Matrices P&Q-

Let A be a non-zero m × n matrix.

Write $A = I_m A I_n$ (which is obviously true).

Reduce A on the L. H. S to normal form by applying elementary row and column transformations on A in such a way that the each elementary row transformation applied on A will be also applied to I_m on R. H. S and each elementary column transformation applied on A will be applied to I_n on R. H. S.

After a sequence of suitable applications of elementary transformations, we get

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

Then the rank of A = the rank of $I_r = r$

SOLVED EXAMPLES

Example 1 Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ to normal form and hence find the rank.

Solution Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix} C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix} \begin{matrix} C_3 \to C_3 + (-2)C_2 \\ C_4 \to C_4 - C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} R_3 \to R_3 - (-3)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_3 \to \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad C_4 \to C_4 - C_3$$

$$= [I_3 : 0]$$

This is the normal form of A and so ρ (A) = 3

Example 2 Find two non-singular matrices P and Q such that PAQ is in the normal form

Where $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. Also find rank of A.

Solution Given $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}_{3\times 3}$

We have $A = I_3 A I_3$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To obtain matrices P & Q, we reduce the LHS matrix to normal form under the fact that

row operations to be applied to pre factor and column operations to be applied to post factor.

Apply, $C_2 \to C_2 + C_1$

$$C_3 \to C_3 + C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 3 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and to post factor

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + (-3)R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And to pre factor

$$R_2 \to \frac{1}{2}R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And to pre factor

$$C_3 \to C_3 - C_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

And to post factor

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

It is the normal form of the given matrix and hence the rank of A i.e.ρ (A) = 2and

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -2 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3 Let $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}$. Find the non-singular matrices P and Q, such that PAQ is in the normal form. Also find the rank of A.

Solution Given $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}_{3\times 4}$

We have $A = I_3 A I_4$

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To obtain matrices P & Q, we reduce the LHS matrix to normal form under the fact that row operations to be applied to pre factor and column operations to be applied to post factor.

$$\begin{matrix} R_2 \to R_2 + (-4)R_1 \\ R_3 \to R_3 + (-2)R_1 \\ and\ to\ pre\ factor \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 6 & -9 \\ 0 & 4 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} C_4 \to C_4 + \frac{3}{2}C_2 \\ and\ to\ post\ factor \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{4}{3} & \frac{1}{3} & 0 \\ -1 & 0 & \frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_3 \to R_3 - R_2 \\ and\ to\ pre\ factor \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_3 : 0] = PAQ,$$

Where $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and the rank of A = 3

Example 4 Find non-singular matrices P , Q so that PAQ is a normal form where

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution Order of A is 3×4

Total number of rows in A = 3 Hence consider unit matrix I_3

Total number of column in A = 4

Hence consider unit matrix I_4

$A_{3\times 4} = I_3 A I_4$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -R_2$, $R_3 \to -R_3$, $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 6 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 6R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & -28 & -56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 6 & -1 & -9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - 5C_2$, $C_4 \to C_4 - 10C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -28 & -56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 6 & -1 & -9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to -\frac{1}{28} R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -\frac{6}{28} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_4 \to C_4 - 2C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -\frac{3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

N = PAQ where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -\frac{3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: P and Q are not unique.

3. By Reducing Into Echelon Form

A matrix is said to be in Echelon form if

(i) the number of zeros preceding the first non-zero element in each row is less than that in the subsequent row(s),

(ii) all zero rows (i.e., rows with zero elements only), if any, are on the bottom of the matrix i.e. all the non- zero rows precede the zero rows,

(iii) the first non-zero element in each row is unity.

For Example-$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & -1 & 0 & 4 & 5 \\ 0 & -1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $and\ D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ are in Echelon forms

Result: If a matrix A is reduced to Echelon form then rank of A i.e.ρ(A) = the number of non-zero rows in its Echelon form

Thus in the above Examples, ρ(A) = 2, ρ(B) = 2, ρ(C) = 3 and ρ(D) = 3.

SOLVED EXAMPLES

Example 1 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$, by reducing to Echelon form.

Solution

Given $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \begin{matrix} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + (-3)R_1 \\ R_4 \to R_4 + (-6)R_1 \end{matrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix} R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} R_4 \to R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} R_4 \to R_4 - R_3$$

which is echelon form

$\therefore \rho$(A) = the number of non-zero rows in the Echelon form = 3

Example 2 Determine the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 6 & 9 & -3 \\ 2 & 4 & 6 & -2 \end{bmatrix}$, by reducing to echelon form.

Solution

Given $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 6 & 9 & -3 \\ 2 & 4 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + (-3)R_1 \\ R_3 \to R_3 + (-2)R_1 \end{matrix}$

= B, which is a row echelon matrix.

$\therefore$ r(A) = the number of non-zero rows in B = 1

Example 3 Find the value of k if the rank of the matrix $\begin{bmatrix} 6 & 3 & 5 & 9 \\ 5 & 2 & 3 & 6 \\ 3 & 1 & 2 & 3 \\ 2 & 1 & 1 & k \end{bmatrix}$ is 3.

Solution $A = \begin{bmatrix} 6 & 3 & 5 & 9 \\ 5 & 2 & 3 & 6 \\ 3 & 1 & 2 & 3 \\ 2 & 1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{6} & \frac{3}{2} \\ 5 & 2 & 3 & 6 \\ 3 & 1 & 2 & 3 \\ 2 & 1 & 1 & k \end{bmatrix} R_1 \to \frac{1}{6} R_1$

$$\sim\begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{6} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{3}{6} & -\frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{3}{6} & -\frac{3}{2} \\ 0 & 0 & -\frac{4}{6} & k-3 \end{bmatrix} \begin{matrix} R_2 \to R_2 + (-5)R_1 \\ R_3 \to R_3 + (-3)R_1 \\ R_4 \to R_4 + (-2)R_1 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{6} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{3}{6} & -\frac{3}{2} \\ 0 & 0 & \frac{4}{6} & 0 \\ 0 & 0 & -\frac{4}{6} & k-3 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim\begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{6} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{3}{6} & -\frac{3}{2} \\ 0 & 0 & \frac{4}{6} & 0 \\ 0 & 0 & 0 & k-3 \end{bmatrix} R_4 \to R_4 + R_3$$

$= B$

Given r(A) = 3. So, the number of non-zero rows of B should be 3.

$\therefore k - 3 = 0 \Rightarrow k = 3$

EXERCISE

Find the rank of the following matrices reducing to echelon form

1. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 3 & 0 & 2 \\ 3 & 4 & -1 & 3 \\ -7 & -7 & 1 & 5 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & 2 & 1 \\ -1 & -1 & -2 & 1 \\ 1 & 2 & 1 & -1 \\ 1 & 3 & 0 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -4 & 6 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{bmatrix}$

10. Find the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$, by reducing to an echelon matrix

11. Find the rank of the matrix $\begin{bmatrix} -2 & -1 & -1 \\ 12 & 8 & 6 \\ 10 & 5 & 6 \end{bmatrix}$

12. Reduce the matrix $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ to normal form and hence find the rank

13. Find the values of k if the rank of $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$ is 3.

14. Find the values of a and b if the matrix $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 4 \\ 7 & -1 & a & b \end{bmatrix}$ is of rank 2.

15. Find the values of a and b if the matrix $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix}$ is of rank 2.

16. Reduce to normal form and find the rank of $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$.

17. Reduce to normal form and find the rank of $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$.

18. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ then find non-singular matrices P and Q such that PAQ is in normal form and find its rank.

19. If A is a non-zero 3×1 matrix and B is a non-zero 1×3matrix , then the rank of AB is:

(a) 0

(b) 1

(c) 2

(d) 3

20. Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is :

(a) An identity matrix (b)A scalar matrix (c)In normal form (d)In echelon form

Answers

1. 3
2. 2
3. 3
4. 2
5. 3
6. 3
7. 3
8. 3
9. 3
10. 2
11. 3
12. 3
13. 2
14. 4,18
15. 4,6
16. $[I_4, 0], rank = 4$
17. $[I_3, 0], rank = 3$
18. $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and rank = 2
19. (b)
20. (d)

CHAPTER- 4

SOLUTIONS OF SYSTEM OF LINEAR EQUATIONS

4.1 Matrix representation of Linear equations

Consider a system of m linear equations in n unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_1xx_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \dots (1)$$

In matrix form , we can write the system of equations (1) as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A X = B

where A , B , X are respectively called the coefficient matrix , requirement vector and the solution vector .

4.2 Non- Homogeneous and Homogeneous Systems of Linear Equations –

The system of equations given by (1) is said to be the system of non-homogeneous linear equations if at least onebi $\neq 0$ i.e, $B \neq O\ (nul\ \ matrix)$ and it is called homogeneous if each b_i= 0, thus matrix form of homogeneous linear equations is $A\,X = 0$.

A Consistent and Inconsistent system A system of equations is called consistent if it has at least one solution otherwise it is called inconsistent. A consistent system has either a unique solution or infinitely many solutions. A homogeneous system is always consistent.

Methods to solve simultaneous linear equations –

The following methods are useful to solve linear equations

1. Matrix Inversion method (or using of adjoint of the coefficient matrix)
2. Cramer`s rule (or using of determinant)
3. Gauss elimination method