CHAPTER- 4

SOLUTIONS OF SYSTEM OF LINEAR EQUATIONS

4.1 Matrix representation of Linear equations

Consider a system of m linear equations in n unknowns

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_1xx_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \dots (1)$$

In matrix form , we can write the system of equations (1) as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A X = B

where A , B , X are respectively called the coefficient matrix , requirement vector and the solution vector .

4.2 Non- Homogeneous and Homogeneous Systems of Linear Equations –

The system of equations given by (1) is said to be the system of non-homogeneous linear equations if at least onebi $\neq 0$ i.e, $B \neq O\ (nul\ \ matrix)$ and it is called homogeneous if each b_i= 0, thus matrix form of homogeneous linear equations is $A\,X = 0$.

A Consistent and Inconsistent system A system of equations is called consistent if it has at least one solution otherwise it is called inconsistent. A consistent system has either a unique solution or infinitely many solutions. A homogeneous system is always consistent.

Methods to solve simultaneous linear equations –

The following methods are useful to solve linear equations

1. Matrix Inversion method (or using of adjoint of the coefficient matrix)
2. Cramer`s rule (or using of determinant)
3. Gauss elimination method

4. Method of rank approach

As per discussion in the previous classes, it is well known that the first two methods are applicable only when $m = n$ i.e. to solve system of n equations in n unknowns (or when the coefficient matrix is a square matrix) and also these methods are not used for large values of n, say $n > 4$, while the third and forth methods are applicable in general i.e. also in the case when $m \neq n$ or $n > 4$.

In the next section, we deal with these method one-by-one .The method of rank approach is the more general method since it can be used in the all situations so we shall discuss it in detailed while the others in brief .

Matrix Inversion method (or using adjoint of the coefficient matrix)-

For Non-Homogeneous system –

Consider a non-homogeneous system of n- equations in n unknowns

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_1xx_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \dots (1)$$

Its matrix form is $A\,X = B$

Where$A = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{nn} \end{bmatrix}$,$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

First find the determinant of the coefficient matrix A i.e. $|A|$.

Now

i. If $|A| \neq 0$then the system is consistent and has unique solution given by

$$X = A^{-1}B$$

ii. If $|A| = 0$ and $(adj\ A)B = O$ then the system is consistent and has infinite many solutions.

iii. If $|A| = 0$ and $(adj\ A)B \neq O$ then the system is inconsistent i.e. has no solution.

For Homogeneous system-

Let $AX = 0$ be a homogeneous system of n-equations in n unknowns.

i. If A is non singular i.e.$|A| \neq 0$ then the system has unique solution

$$x_1 = 0, x_2 = 0, \quad \dots \dots . \dots, x_n = 0$$

This solution is called zero or trivial solution

ii. If A is singular i.e.$|A| = 0$ then the system has also non trivial (non-zero) solutions

Example 1 Test the consistency of the following system of equations and if consistent find the solution

$$2x - y + 3z = 9$$

$$x + y + z = 6$$

$$x - y + z = 2$$

Solution The matrix form the given equations is

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

i.e. $\quad AX = B$

we have $|A| = 2(1+1) + 1(1-1) + 3(-1-1)$

$$= 2.2 + 1.0 + 3(-2) = -2 \neq 0$$

Since $|A| \neq 0$ therefore the given system of equations is consistent and has unique solution given by

$$X = A^{-1}B \qquad \dots (1)$$

Now to obtain A^{-1} we have

Cofactor of $a_{11} = c_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$

And similarly we can find $c_{12} = 0, c_{13} = -2, c_{21} = -2, c_{22} = -1, c_{23} = 1, c_{31} = -4, c_{32} = 1, c_{33} = 3$

$$\text{adj A} = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

and$A^{-1} = \frac{1}{|A|}\, adj\ A = -\frac{1}{2} \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$

Now by (1), we have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Hence the required solution of the given section the equations, is $\quad x = 1, y = 2, z = 3$

Example 2 using matrix method test the consistency of the equations

$$3x - y - 2z = 2$$

$$2y - z = -1$$

$$3x - 5y = 3$$

Solution The matrix form of the given system is $\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

A X = B

We have

$|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ = 3 (0 - 5) + 1 (0 + 3) + 1 (0 + 3) – 2 (0 - 6) = -15 + 3 + 12 = -15 + 15 = 0

And the conations of the elements of A are

C_{11} = -5 C_{12} = -3 C_{13} = -6

$C_{21} = 5$ $C_{22} = 6$ $C_{23} = 12$

$C_{31} = 5$ $C_{32} = 3$ $C_{33} = 6$

$$\therefore \text{ adj A} = \begin{bmatrix} -5 & -3 & -6 \\ 5 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}$$

now

$$(\text{adj A}) \text{ B} = \begin{bmatrix} -5 & -3 & -6 \\ 5 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 & +3 & -18 \\ 10 & -6 & +36 \\ 10 & -3 & +18 \end{bmatrix} = \begin{bmatrix} -25 \\ 30 \\ 25 \end{bmatrix}$$

Since $|A| = 0$ and (adj A) B$\neq 0$ therefore the system is in consistent i, e. has no solution .

Example 3 Determine the values of K for which the system of equations

x – xy + z = 0

kx + 3y – kz = 0

3x + y – z = 0

Has (i) only trivial solution (ii) non - trivial solution.

Solution – The given system is homogeneous and its matrix form is

Ax = 0

Where

$$A = \begin{bmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{bmatrix} \; \& \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We have $|A| = \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix}$

= 1 (-3 + k) + k (-k + 3k) + 1 (k - 9)

= -3 + k + $2k^2$ + k – 9

= $2k^2$ + 2k – 12 = 2 (k- 3) (k + 4)

We know that the given has only trivial solution if $|A| \neq 0$

= 2 (k - 3) (k + 4) $\neq$ 0 = k $\neq$ 3 & k $\neq$ - 4

And for non trivial solution ,$|A|$ = 0

= k = 3 or k = -4 .

CRAMER'S RULE (USING DETERMINATS)

For Non Homogeneous system

As mentioned earlier that the matrix invention method and Cramer s rule are not used for large values of n , say n >4, i. e, not to solve system of move than 4 ×4 order , so we have consider a system of 3 equations in 3 unknown ,

$$a_1\, x \;+\; b_1\, y \;+\; c_1\, z \;=\; d_1$$
$$a_2\, x \;+\; b_2\, y \;+\; c_2\, z = \; d_2$$
$$a_3\, x \;+\; b_2\, y \;+\; c_3\, z = \; d_3$$

Find we tied the values of the values of the following four detriments

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c\_3 \end{vmatrix} \qquad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c\_3 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now

i. If $D \neq 0$ then system is consistent and has unique solution given by

$x = \frac{D_1}{D}$ $\qquad y = \frac{D_2}{D}$ $\qquad z = \frac{D_3}{D}$

ii. If $D = 0$ and $D_1 = D_2 = D_3 = 0$ then system is consistent and has infinitely many solutions.

iii. If $D = 0$ and none of $D_1, D_2 \& D_3$ is zero then system is inconsistent i,e. has no solution .

For non- Homogeneous System -

In this care there are the same conclusions as mentioned in the previous method for homogeneous system

Example 1 using Cramer's rule, solve

$$x_1 - x_2 + 3x_3 = 3$$

$$2x_1 + 3x_2 + x_3 = 2$$

$$3x_1 + 2x_2 + 4x_3 = 5$$

Solutions we have

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 1(12-2) + 1(8-3) + 3(4-9) = 15 - 15 = 0$$

$$D_1 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = 3(12-2) + 1(8-5) + 3(4-15) = 30 + 3 - 33 = 0$$

$$D_2 = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 0 \quad \& D_3 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0$$

$\because$ $D = 0$ and $D_1 = D_2 = D_3 = 0$ therefore the given system is consistent and has infinite number of solutions which can be obtained as :

Assume $z = k$ where k is an orbiting constant

Now putting $z = k$ in the two equations and solving for $x \,\&\, y$ we get.

$$x = \frac{11-10k}{5} \qquad y = \frac{5k-4}{5}$$

Example 2. using Cramer's rule solve

$$x - y + z = 4$$

$$2x + y - 3z = 0$$

$$x + y + z = 2$$

Solution- we have

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1+3) + 1(2+3) + 1(2-1) = 10$$

$$D_1 = \begin{vmatrix} 4 & -1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 4(1+3) + 1(0+6) + 1(0-2) = 20$$

$$D_2 = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 1(0+6) - 4(2+3) + 1(4-0) = -10$$

$$D_3 = \begin{vmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-0) + 1(4-0) + 4(2-1) = 10$$

Since $D \neq 0$ therefore the given system has unique solve as

$$x = \frac{D_1}{D} = \frac{20}{10} = 2, y = \frac{D_2}{D} = \frac{-10}{10} = -1 \;\&\; z = \frac{D_3}{D} = \frac{10}{10} = 1 .$$

GAUSS ELIMINATION METHOD

Consider the system of equations $AX = B$ ….(1)

In this method, apply row elementary transformation for matrix on both A and B so that the matrix A is reduced to the upper triangular form. if in this process to A, and B to B, then AX= B given $A_1X = B,$ which when solved by back substitution gives the required solution of the given system of equations …(1) .

Example 1 solve the system

$$x - 2y + 3z = 2$$
$$2x - 3z = 3$$
$$x + y + z = 0$$

By Gauss elimination method

Solution The matrix form of the given system is $\begin{bmatrix}1 & -2 & 3\\2 & 0 & -3\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\3\\0\end{bmatrix}$

A X = B

By applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$\begin{bmatrix}1 & -2 & 3\\0 & 4 & -9\\0 & 3 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\-1\\-2\end{bmatrix}$$

By applying $R_3 \to R_3 - \frac{3}{4}\ R_2$

$$\begin{bmatrix}1 & -2 & 3\\0 & 4 & -9\\0 & 0 & \frac{19}{4}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\-1\\\frac{-5}{4}\end{bmatrix}$$

Example 2 By Gauss elimination method, solve

$$x_1 + x_2 + 2x_3 + x_4 = 5$$
$$2x_1 + 3x_2 - x_3 - 2x_4 = 2$$
$$4x_1 + 5x_2 + 3x_3 = 7$$

Solution the matrix form of the system is

$$\begin{bmatrix}1 & 1 & 2 & 1\\2 & 3 & -1 & -2\\4 & 5 & 3 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}5\\2\\7\end{bmatrix}$$

By $R_2 \to R_2 - 2R_2 \& R_3 \to R_3 - 4R_1$

$$\begin{bmatrix}1 & 1 & 2 & 1\\0 & 1 & -5 & -4\\0 & 1 & -5 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}5\\-8\\-13\end{bmatrix}$$

By $R_3 \to R_3 - R_2$

$$\begin{bmatrix}1 & 1 & 2 & 1\\0 & 1 & -5 & -4\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}5\\-8\\-13\end{bmatrix}$$

$$\Rightarrow x_1 + x_2 + 2x_3 + x_4 = 5$$

$$0x_1 + 1x_2 - 5x_3 - 4x_4 = -8$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -5$$

Then last one of these equations show that $0 = -5,$ which is absurd. Hence the given system of equations is inconsistent.

METHOD OF RANK APPROACH

For non Homogeneous system

Consider a non homogeneous system of m equations in n unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_1xx_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \ldots (1)$$

The matrix form, of the system, is

$$A\,X = B$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ called coefficient matrix.

Write the augmented matrix

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & \vdots & b_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & \vdots & b_m \end{bmatrix}$$

And then find the ranks of the coefficient matrix A and augmented matrix [A, B] for which, reduce [A, B] to echelon form by applying row elementary transformations only. This echelon form gives the rank of the augmented matrix [A , B] and by deleting the last column of the echelon form of [A , B] we get the echelon form of the coefficient matrix A which gives the rank of A. Thus the echelon form of the augmented matrix [A , B] provides the ranks of A & [A,B] both.

Now there may arise any one of the following cases

i. If ρ(A) = ρ[A , B] = r = n (no. of unknowns) then system is consistent and has unique solution.
ii. If ρ(A) = $\rho[A, B] = r < n$ (no. of unknowns) then system is consistent and has infinite number of solutions in this case only $(n - r + 1)$ solutions are linearly independent and rest of the solutions are the liners combinations of there $(n - r + 1)$ independent solutions.
iii. If $\rho(A) \neq \rho[A, B]$ then system is inconsistent i.e. has no solution.

For homogenous system –

Consider a homogeneous system of m equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{12}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{22}x_n = 0$$
$$-----------------$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

The matrix form is AX = 0

Where A = $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

Some Important Results about the Nature of Solutions of the Equation AX = O.

Suppose we have m equations in n unknowns. Then the coefficient matrix A will be of the type m x n. Let r be the rank of the matrix A. Obviously r cannot be greater than n (the number of columns of the matrix A). Therefore we have either r = n or r<n.

Case I. If r = n, the equation AX = O will have n –n i.e., no linearly independent solutions. In this case the zero solution will be the only solution. We know that zero vector forms a linearly dependent set.

Case II. If r < n, we shall n – r linearly independent solutions. Any linear combination of these n – r solutions will also be a solution of AX = O. Thus in this case the equation AX = O will have an infinite number of solutions.

Case III. Suppose m < n i.e., the number of equations is less than the number of unknowns. Since r $\leq$ m, therefore r is definitely less than n. Hence in this case the given system of equations must possess a non-zero solution. The number of solutions of the equation AX = O will be infinite.

Remark

1. If the coefficient matrix is a square matrix i.e. no. of equations is equal to no. of unknowns then system has only trivial solution $if\ |A| \neq 0$. And also non trivial solutions if $|A| = 0$

2. A system of homogenous linear equations has a non trivial solution if the number of equations is less than the number of unknowns.

Fundamental Set of Solutions of the Equation AX = O.

Suppose the rank r of the coefficient matrix A is less than the number of the unknown's n. In this case the given equations have a set of n – r linearly independent solutions and every possible solution is a linear combination of these n – r solutions. This set of n – r solutions is called a fundamental set of solutions of the equation AX = O.

A set of linearly independent solution $X_1, X_2, \ldots., X_k$ of the system of homogeneous equations AX = O is called the fundamental system of solutions of AX = O, if every solution X of AX = O can be written as a linear combination of these i.e., in the form

$$X = c_1X_1 + c_2X_2 + \ldots\ldots. + c_kX_k,$$

Where $c_1, c_2, \ldots\ldots., c_k$ are suitable numbers.

SOLVED EXAMPLES

Example 1 show that system of equations

$$3x + 3y + 2z = 1$$

$$x + 2y = 4$$

$$10y + 3z = -2$$

$2x - 3y - z = 5,$ is consistent and hence solve it.

Solution The given of equation can be written in the matrix form

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 1 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

$$AX = B$$

The augmented matrix$[A, B] = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$

$$R_1 \leftrightarrow R_2$$

$$[A, B] \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix} R_2 \longrightarrow R_2 - 3R_1 R_4 \longrightarrow R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & \frac{29}{3} & -\frac{116}{3} \\ 0 & 0 & -\frac{17}{3} & -\frac{68}{3} \end{bmatrix} R_3 \longrightarrow R_3 + 10R_2 R_4 \longrightarrow R_4 \frac{-7}{3} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{bmatrix} by\ R_3 \longrightarrow \frac{3}{29} R_3 R_4 \longrightarrow R_4 \longrightarrow \frac{-3}{17} R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_4 \longrightarrow R_4 - R_3$$

$$\Rrightarrow \rho([A.B]) = 3$$

And also $\rho(A) = 3$

i.e. $\rho([A,B]) = \rho(A) = 3 = N0.\,of$ unknowns therefore the given system is consistent and has unique solution

Further, from the above echelon form, the given system in matrix form reduces to

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -4 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + 2y = 4, -3y + 2z = -11 \& z = -4$$

$$\Rightarrow x = 2, y = 1, z = -4$$

Thus the unique solution is $x = 2,\ y = 1,\ z = -4$

Example 2 Investigate for what values of $\lambda, \mu,$ the equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

Have (1) no solution, (2) a unique solution, (3) infinity of solutions.

Solution The given system of equations can be written in the following matrix form :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Performing the operations $R_{21}(-1)$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu - 6 \end{bmatrix}$$

Now performing $R_{32}(-1)$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu - 10 \end{bmatrix}$$

Whence $(\lambda - 3)z = \mu - 10$.

We thus observe the following :

i. If $\lambda = 3, \mu \neq 10,$ the system is inconsistent, so there is no solution.

ii. If $\lambda \neq 3,$ it is possible to find a unique value of z and then unique values of x and y. in this case there is a unique solution.

iii. If $\lambda = 3, \mu = 10$, the last equation has the form $0.\ z = 0$ and we can assign arbitrary values to z, obtaining corresponding values of x and y. So in this case, the number of solutions is infinity .

Example 3 Investigate for what values of $\lambda\ and\ \mu$ the equations

$$x + y + z = 6,$$

$$x + 2y + 4y = 10,$$

$$2x + 3y + \lambda z = \mu$$

Have (1) no solution, (2) a unique solution, and (3) infinitely many solutions.

Solution The given system of equations can be written in the following matrix form :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Performing the operations $R_{31}(-2)$ and R_{31}, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu - 12 \end{bmatrix}$$

Now performing $R_{32}(-1)$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu - 16 \end{bmatrix}$$

Whence $(\lambda - 5)z = \mu - 16$.

We thus observe the following :

i. If $\lambda = 5, \mu \neq 16,$ the system is inconsistent, so there is no solution.

ii. If $\lambda \neq 5,$ it is possible to find a unique value of z form (1), and then unique values of x and y. So there is a unique solution.

iii. If $\lambda = 5, \mu \neq 16,$ equation (1) has the form 0. z = 0 and we can assign arbitrary values to z, obtaining corresponding values of x and y. So, the number of solution is infinity.

Example 4 Find out for what values of λ, the equations

$$x + y + z = 1,$$

$$x + 2y + 4z = \lambda,$$

$$x + 4y + 10z = \lambda^2$$

Have a solution and solve completely in each case.

Solution Here we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \; and \; [A, B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 1 & 2 & 4 & \vdots & \lambda \\ 1 & 4 & 10 & \vdots & \lambda^2 \end{bmatrix}$$

By the operations $R_{21}(-1)$ and $R_{31}(-1)$, we have

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 3 & \vdots & \lambda - 1 \\ 0 & 3 & 9 & \vdots & \lambda^2 - 1 \end{bmatrix}$$

By $R_{32}(-3)$, we have

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 3 & \vdots & \lambda - 1 \\ 0 & 0 & 0 & \vdots & \lambda^2 - 3\lambda + 2 \end{bmatrix} \quad \ldots\ldots.(1)$$

Whence rank A=2. The system of equations will be consistent if rank $[A, B] = rank\ A = 2$. Hence we must have

$$\lambda^2 - 3\lambda + 2 = 0, \qquad so\ that\ \ \lambda = 1, or\ 2.$$

From (1), an equivalent system of equations is

$$x + y + z = 1,$$

$$y + 3z = \lambda - 1. \quad \text{.......(2)}$$

Since rank A= rank $[A, B] = r = 2 < n = 3$ (the number of unknowns),there will be an infinite number of solutions. Taking $z = k$, where k is an arbitrary constant, and solving the equations (2), we obtain

$$x = 2k - \lambda + 2,$$

$$y = \lambda - 1 - 3k,$$

$$z = k.$$

And

When $\lambda = 1$ the solutions are given by $x = 2k + 1, y = -3k, z = k$.

When $\lambda = 2$, the solutions are given by $x = 2k, y = 1 - 3k, z = k$.

Example 5 How many number of linearly independent solutions of the following equations are possible?

$$x_1 + x_2 - 2x_3 + x_4 + 3x_5 = 1,$$

$$2x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 = 2$$

$$3x_1 + 2x_2 - 4x_3 - 3x_4 - 9x_5 = 3.$$

Solution The augmented matrix:

$$[A, B] = \begin{bmatrix} 1 & 1 & -2 & 1 & 3 & : & 1 \\ 2 & -1 & 2 & 2 & 6 & : & 2 \\ 3 & 2 & -4 & -3 & 9 & : & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 1 & 3 & : & 1 \\ 0 & -3 & 6 & 0 & 0 & : & 0 \\ 0 & -1 & 2 & -6 & -18 & : & 0 \end{bmatrix}, \text{ by } R_{21}(-2) and R_{31}(-3)$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 1 & 3 & : & 1 \\ 0 & 1 & -2 & 0 & 0 & : & 0 \\ 0 & -1 & 2 & -6 & -18 & : & 0 \end{bmatrix}, \text{by } R_2\left(-\frac{1}{3}\right)$$

$$\sim\begin{bmatrix}1 & 1 & -2 & 1 & 3 & : & 1\\ 0 & 1 & -2 & 0 & 0 & : & 0\\ 0 & 0 & 0 & -6 & -18 & : & 0\end{bmatrix}, \text{ by } R_{32}(1)$$

$$\sim\begin{bmatrix}1 & 1 & -2 & 1 & 3 & : & 1\\ 0 & 1 & -2 & 0 & 0 & : & 0\\ 0 & 0 & 0 & 1 & 3 & : & 0\end{bmatrix}, \text{ by } R_3\left(-\frac{1}{6}\right).$$

This matrix is in echelon form. The number of non-zero rows in it for both the matrices A and $[A, B]$ is 3. Hence

Rank A= rank $[A, B] = 3$ $(= r, say)$.

thus the given system of equations is consistent.

Also, the number of linearly independent solution

$$= n - r + 1 = 5 - 3 + 1 = 3.$$

Example 6 Find the real values of λ for which the following equations have a non-zero solution :

$$x + 2y + 3z = \lambda x,$$

$$3x + y + 2z = y\lambda,$$

$$2x + 3y + z = \lambda z\,.$$

Solution The given equations can be written as

$$(1 - \lambda)x + 2y + 3z = 0,$$

$$3x + (1 - \lambda)y + 2z = 0,$$

$$2x + 3y + (1 - \lambda)z = \lambda z\,.$$

We know that a system of n linear homogeneous equations in n unknowns has a non-zero solution if the coefficient matrix is singular.

In the present case, we have

Number of equations = 3 = number of unknown.

So, the above equations will have a non-zero solution if

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0,$$

i.e., $(1-\lambda)\{(1-\lambda)^2 - 6\} + 2\{4 - 3(1-\lambda)\} + 3\{9 - 2(1-\lambda)\} = 0,$

i.e., $(1-\lambda)(\lambda^2 - 2\lambda - 5) + 2(1 + 3\lambda) + 3(7 + 2\lambda) = 0,$

i.e., $(\lambda^2 - 2\lambda - 5 - \lambda^3 + 2\lambda^2 + 5\lambda) + 23 + 12\lambda = 0,$

i.e., $\lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0,$

i.e., $(\lambda - 6)(\lambda^2 + 3\lambda + 3) = 0.$

The only real value of λ obtained from this is 6.

Hence 6 is the only real value of λ for which the given equations have a non-zero solution .

Example 7 Solve completely the system of equations

$$x - 2y + z - w = 0,$$

$$x + y - 2z + 3w = 0,$$

$$4x + y - 5z + 8w = 0,$$

$$5x - 7y + 2z - w = 0\,.$$

Solution The given system can be written in the following matrix form:

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

To reduce the coefficient matrix to Echelon to form, we first perform $R_{21}(-1), R_{31}(-4) and\ R_{41}(-5)$. Thus we have

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Next performing $R_{32}(-3) and\ R_{42}(-1)$, we get

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, performing $R_2\left(\frac{1}{3}\right)$, we obtain

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Where the coefficient matrix is in echelon form.

This matrix equation is equivalent to the equations

$$x - 2y + z - w = 0,$$

$$y - z + \frac{4}{3}w = 0\,.$$

We may now assign any values to z and w. Let us put $z = c_1$ and$w = c_2$. Then the equations reduce to

$$x - 2y = -(c_1 - c_2),$$

$$y = c_1 - \frac{4}{3}c_2.$$

Putting this value of y in the first one of these equations, we get $x = c_1 - \frac{5}{3}c_2$. Hence

$$x_1 = c_1 - \frac{5}{3}c_2, y = c_1 - \frac{4}{3}c_2, z = c_1, w = c_2$$

Is the general solution of the given system, where $c_1\ and\ c_2$ are arbitrary numbers.

4.3 Linear Dependence and Linear Independence of Vectors.

A set of n vectors X_1,,X_2 , X_3,…….A_n are said to be linearly dependent(L.D.) if there exist n scalars α_1, α_2, α_3……..α_n , not all are zero, such that

$$\alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3 + \ldots\ldots + \alpha_nX_n = O$$

further a set of n vectors $X_1, X_2, X_3, \ldots\ldots X_n$ are said to be linearly independent(L.I.) if there exist n scalars $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_n$, all are zero, such that

$$\alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3 + \ldots\ldots + \alpha_nX_n = O$$

Important Facts-

- **i.** A singleton set consisting of a non-zero vector is always L.I. and a singleton set consisting of a zero vector is always L.D.
- **ii.** Two vectors are L.D. if and only if one of them can be expressed as a scalar multiple of the other.
- **iii.** If in a set of vectors, any vector of the set is the linear combination of the remaining vectors, then the vectors are L.D.

Linear Dependence and Linear Independence of Vectors by Rank Method-

- **i.** If the rank of the matrix of the given vectors is equal to number of vectors, then the vectors are linearly independent.
- **ii.** If the rank of the matrix of the given vectors is less than the number of vectors, then the vectors are linearly dependent.

SOLVED EXAMPLES

Example 1 Show using a matrix that the set of vectors

$X = [1 \quad 2 \quad -3 \quad 4]$, $Y = [3 \quad -1 \quad 2 \quad 1]$, $Z = [1 \quad -5 \quad 8 \quad -7]$ is linearly dependent.

Solution Here we have

$X = [1 \quad 2 \quad -3 \quad 4]$, $Y = [3 \quad -1 \quad 2 \quad 1]$, $Z = [1 \quad -5 \quad 8 \quad -7]$

Let us form a matrix of the above vectors

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 1 \\ 1 & -5 & 8 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & -7 & 11 & -11 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the rank of matrix = 2 < number of vectors

Hence , vectors are linearly dependent.

Example 2 Show using a matrix that the set of vectors : [2 5 2 -3] , [3 6 5 2] ,

[4 5 14 14] , [5 10 8 4] are linearly independent.

Solution The given vectors are

[2 5 2 -3] , [3 6 5 2] , [4 5 14 14] , [5 10 8 4]

Let us form a matrix of the above vectors

$$A = \begin{bmatrix} 2 & 5 & 2 & -3 \\ 3 & 6 & 5 & 2 \\ 4 & 5 & 14 & 14 \\ 5 & 10 & 8 & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_3$

$$\sim\begin{bmatrix} 2 & 5 & 2 & -3 \\ 1 & 1 & 3 & 5 \\ 1 & -1 & 9 & 12 \\ 1 & 5 & -6 & -10 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim\begin{bmatrix} 1 & 1 & 3 & 5 \\ 2 & 5 & 2 & -3 \\ 1 & -1 & 9 & 12 \\ 1 & 5 & -6 & -10 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\sim\begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 3 & -4 & -13 \\ 0 & -2 & 6 & 7 \\ 0 & 4 & -9 & -15 \end{bmatrix}$$

, $R_3 \to R_3 + \frac{2}{3}R_2$, $R_4 \to R_4 - \frac{4}{3}R_2$

$$\sim\begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 3 & -4 & -13 \\ 0 & 0 & \frac{10}{3} & \frac{-5}{3} \\ 0 & 4 & \frac{-11}{3} & \frac{7}{3} \end{bmatrix}$$

$$R_4 \to R_4 + \frac{11}{10}R_3$$

$$\sim\begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 3 & -4 & -13 \\ 0 & 0 & \frac{10}{3} & \frac{-5}{3} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Here the rank of the matrix = 4 = Number of vectors.

Hence the vectors are linearly independent.

EXERCISE

1. Solve the following equations using matrix methods

$$x + y + z = 9$$

$$2x + 5y + 7z = 52$$

$$2x + y - z = 0$$

2. Examine the consistency of the following systems of equations, and if consistent find the complete solutions:

$$2x + 4y - z = 9,$$

$$3x - y + 5z = 5,$$

$$8x + 2y + 9z = 19.$$

3. Apply the test of rank to show that the following systems of equations are consistent. Also solve them.

(i)

$$x + 2y - 5z = -9,$$

$$3x - y + 2z = 5,$$

$$2x + 3y - z = 3,$$

$$4x - 5y + z = -3.$$

(ii)

$$x + 2y - z = 3,$$

$$3x - y + 2z = 1,$$

$$2x - 2y + 3z = 2,$$

$$x - y + z = -1$$

4. Test the consistency of the following systems of equations and solve them:

(i)

$$5x + 7y - 13 = 0,$$

$$-3x + 11y - 53 = 0,$$

$$x - 5y - 23 = 0,$$

$$2x - 3y + 18 = 0.$$

5. Show that the following systems of equations are consistent and solve them :

(i) $$x + y + z = 3,$$
$$x + 2y + 3z = 4,$$
$$x + 4y + 9z = 6.$$

(ii) $$4x + 3y + 6z = 25,$$
$$x + 5y + 7z = 13,$$
$$2x + 9y + z = 1.$$

6. Solve the following equations by matrix method :

$$\lambda x + 2y - 2z - 1 = 0,$$
$$4x + 2\lambda y - z - 2 = 0,$$
$$6x + 6y + \lambda z - 3 = 0.$$

considering specially the case when $\lambda = 2$

7. Find out the solutions of following systems of equations :

$$x_1 - 2x_2 + x_3 - x_4 + 1 = 0,$$

$$3x_1 - 2x_3 + 3x_4 + 4 = 0,$$
$$5x_1 - 4x_2 + x_4 + 3 = 0.$$

8. solve the following system of equations :

$$x + y + z = \lambda,$$
$$\alpha x + \beta y + \gamma z = \lambda^2,$$
$$\alpha^2 x + \beta^2 y + \gamma^2 z = \lambda^3.$$

9. Does the following systems of equations possess non-trivial solutions :

(i) $$2x - 3y + z = 0,$$
$$x + 2y - 3z = 0,$$
$$4x - y - 2z = 0.$$

(ii) $$2x + y - 3z = 0,$$
$$x - 3y + 2z = 0,$$
$$4x + 2y - z = 0.$$

10. Find all the solutions of the following systems of equations:

(i)

$$2x - 3y + z = 0,$$

$$x + 2y - 3z = 0,$$

$$4x - y - 2z = 0.$$

(ii)

$$x + 2y + 3z = 0,$$

$$3x + 4y + 4z = 0,$$

$$7x + 6y + 12z = 0.$$

(iii)

$$x_1 - 2x_2 + x_3 = 0,$$

$$x_1 - 2x_2 - x_3 = 0,$$

$$2x_1 - 4x_2 - 5x_3 = 0.$$

(iv)

$$x_1 + x_2 - 6x_3 = 0,$$

$$-3x_1 + x_2 + 2x_3 = 0,$$

$$x_1 - x_2 + 2x_3 = 0.$$

11. Find the general solution of the matrix equation:

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -2 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 0 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0.$$

12. Show, by considering the rank of an appropriate matrix, that the following system of equations possesses no solution other than the trivial solution $x = 0,\ y = 0,\ z = 0$:

$$3x - y + z = 0,$$

$$-15x + 6y - 5z = 0,$$

$$5x - 2y + 2z = 0.$$

13. Let AX = 0 be a system of linear equations, where A = $[\,a_{ij}]_{m\times n}$, X is a column matrix and 0 is a null matrix. If Rank(A) = r <n , then the system has :

(a) Exactly r linearly independent solutions

(b) Exactly n linearly independent solutions

(c) Exactly n - r linearly independent solutions

(d) None of these

14. If two vectors v_1 and v_2 are equal , then the set { v_1 , v_2 , ….., v_n } is:

(a) Linearly dependent

(b) Linearly independent

(c) May not be linearly dependent

(d) None of these

15. The vector $e_1 = (1\ \ 0\ \ 0)$, $e_2 = (0\ \ 1\ \ 0)$ and $e_3 = (0\ \ 0\ \ 1)$ are :

(a) Linearly dependent

(b) Linearly independent

(c) Not linearly independent

(d) None of these