Application of Hankel Transform

Before discussing the application of Hankel transform to solve B.V.P. we derive the following formula

$$\boxed{H_n\left\{ \frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f(x)\right\} = -p^2 H_n\{f(x)\}} \quad \text{—(I)}$$

we have

$$H_n\left\{\frac{d^2f}{dx^2}\right\} = \int_0^\infty \frac{d^2f}{dx^2}\, x J_n(px)\, dx$$

$$= \left|\frac{df}{dx}\, x J_n(px)\right|_0^\infty - \int_0^\infty \frac{df}{dx}\frac{d}{dx}\{x J_n(px)\}dx \quad \text{by parts}$$

$$= -\int_0^\infty \frac{df}{dx}\{x p J_n'(px) + J_n(px)\}dx$$

Assuming $x f'(x) \to 0$ as $x \to 0$ & $x \to \infty$

$$= -p\int_0^\infty \underset{\text{I}}{\frac{df}{dx}}\, \underset{\text{II}}{\underline{x J_n'(px)}}\, dx - \int_0^\infty \frac{1}{x}\frac{df}{dx}\, x J_n(px)\, dx$$

$$= -p\left[f(x)\, x J_n'(px) - \int_0^\infty f(x)\frac{d}{dx}\left(x J_n'(px)\right)\right] - H_n\left\{\frac{1}{x}\frac{df}{dx}\right\}$$

Assuming $x f(x) \to 0$ as $x \to 0$ & $x \to \infty$

$$\Rightarrow H_n\left\{\frac{d^2f}{dx^2}\right\} + H_n\left\{\frac{1}{x}\frac{df}{dx}\right\} = p\int_0^\infty f(x)\frac{d}{dx}[x J_n'(px)]\,dx \quad \text{—(A)}$$

from Bessel's differential equation we have

$$\frac{d}{dx}\left\{x\frac{dy}{dx}\right\} = -\left(1 - \frac{n^2}{x^2}\right) xy = 0$$

This gives $\displaystyle \frac{d}{dx}\{x J_n'(px)\} = \frac{n^2}{px} J_n(px) - px J_n(px)$

using it in (A) we get

$$H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right\} = p\int_0^\infty \frac{n^2}{px} f(x) J_n(px)\, dx - p\int_0^\infty px f(x) J_n(px)\, dx$$

$$= n^2\int_0^\infty \frac{1}{x^2} f(x)\cdot x J_n(px)\, dx - p^2\int_0^\infty x f(x) J_n(px)\, dx$$

(2)

$$H_n\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\} = n^2 H_n\left\{\frac{1}{x^2}f(x)\right\} - p^2 H_n\{f(x)\}$$

i.e.

$$\boxed{H_n\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}f(x)\right\} = -p^2 H_n\{f(x)\}}$$

In particular, $n=0$

$$\boxed{H_0\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\} = -p^2 H_0\{f(x)\}} \quad \text{— (II)}$$

when $n=1$

$$H_1\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{1}{x^2}f(x)\right\} = -p^2 H_1\{f(x)\}$$

APPLICATION of Infinite Hankel Transform -

Ex. 1 Solve $\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=0 \quad r\geq 0,\ z>0$ — (1)

satisfying the conditions

(i) $V\to 0$ as $r\to\infty$ & $z\to\infty$

(ii) $V=f(r)$ on $z=0$, $r\geq 0$

Soln: Taking zeroth order Hankel Transform of (1)

$$H_0\left\{\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right\} + H_0\left\{\frac{\partial^2 V}{\partial z^2}\right\}=0$$

or $$-p^2 H_0\{V\} + \int_0^\infty \frac{\partial^2 V}{\partial z^2}\, r J_0(pr)\,dr = 0$$

[by using above result (II)]

$$\Rightarrow -p^2 H_0\{V\} + \frac{d^2}{dz^2}\int_0^\infty V\, r J_0(pr)\,dr = 0$$

$$\Rightarrow -p^2 H_0\{V\} + \frac{d^2}{dz^2} H_0\{V\} = 0$$

or $$-p^2\bar{V} + \frac{d^2\bar{V}}{dz^2} = 0 \quad \text{where } \bar{V} = H_0\{V\}$$ — (2)

Its soln is $\bar{V} = Ae^{pz} + Be^{-pz}$

From given condition (i) $V \to 0$ as $z \to \infty$

then $\bar{V} \to 0$ as $z \to \infty$ and so $A = 0$

$$\therefore \bar{V} = B e^{-pz} \quad \text{--- (3)}$$

and $(\bar{V})_{z=0} = B$

i.e. $B = (\bar{V})_{z=0} = H_0\{V(r,0)\}$

$$= \int_0^\infty V(r,0)\, r\, J_0(pr)\, dr$$

$$= \int_0^\infty f(r)\, r\, J_0(pr)\, dr$$

$$= H_0\{f(r)\} = \bar{f}_0(p), \text{ say,}$$

$\therefore$ from (3) we have

$$\bar{V} = \bar{f}_0(p)\, e^{-pz} \quad \text{--- (4)}$$

Applying Hankel Inversion formula we obtain

$$V = \int_0^\infty p\, \bar{f}_0(p)\, e^{-pz} J_0(pr)\, dp$$

This is the required soln. of the given B.V.P.

<u>Ex-2</u> Find the potential $V(r,z)$ of a field due to a flate circular disc with its centre at the origin and radius equal to one. The axis of disc is along z-axis and it satisfies the differential equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad 0 \le r < \infty, \; z \ge 0$$

and the boundary conditions

$$V = V_0 \quad \text{when } z = 0,\; 0 \le r < 1$$

$$\frac{\partial V}{\partial z} = 0 \quad \text{when } z = 0,\; r > 1$$

Soln. The given differential eqn is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{--- (1)}$$

Taking zeroth order Hankel transform, we get

$$H_0\left\{\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right\} + H_0\left\{\frac{\partial^2 V}{\partial z^2}\right\} = 0$$

$$\Rightarrow -p^2 \bar{V} + \frac{d^2 \bar{V}}{dz^2} = 0 \quad \text{--- (2)}$$ (by using standard result (II) and let $H_0\{V\} = \bar{V}$)

Its soln is

$$\bar{V} = A e^{pz} + B e^{-pz}$$

Since V is bounded so $\bar{V}$ is also bounded and therefore $A = 0$

and $$\bar{V} = B e^{-pz} \quad \text{--- (3)}$$

Here $B = B(p)$ i.e. B is f^n of p.

Applying inversion formula

$$V = \int_0^\infty B e^{-pz} p J_0(pr)\, dp \quad \text{--- (4)}$$

$$\Rightarrow \frac{\partial V}{\partial z} = \int_0^\infty B e^{-pz} (-p^2) J_0(pr)\, dp \quad \text{--- (5)}$$

putting $z = 0$ in (4) & (5) we get

$$\int_0^\infty p B J_0(pr)\, dp = (V)_{z=0} = V_0 \quad 0 \le r < 1$$

and $$\int_0^\infty -p^2 B J_0(pr)\, dp = \left(\frac{\partial V}{\partial z}\right)_{z=0} = 0 \quad r > 1$$ (by given conditions)

i.e.

$$\left.\begin{aligned} &\int_0^\infty p B J_0(pr)\, dp = V_0 \quad 0 \le r < 1 \\ &\int_0^\infty (-p^2) B J_0(pr)\, dp = 0 \quad r > 1 \end{aligned}\right\} \quad \text{--- (6)}$$

Comparing (6) with the well known integrals

$$\int_0^\infty \frac{\sin p}{p} J_0(pr)\,dp = \pi/2 \qquad 0<r<1$$

$$\int_0^\infty \sin p \, J_0(pr)\,dp = 0 \qquad r>1$$

We obtin $B = \frac{2}{\pi} V_0 \frac{\sin p}{p^2}$

putting this value of B in (4) we obtain

$$V(r,z) = \frac{2V_0}{\pi} \int_0^\infty e^{-pz} \frac{\sin p}{p} J_0(pr)\,dp$$

Ans.

Ex 3. Heat is supplied at a constant rate Q per unit area per unit time over a circular area of radius a in the plane z=0 to an infinite solid of conductivity K. Show that the steady temperature at a distant r from the axis of circular area and distant z from the plane z=0 is given by

$$\frac{Qa}{2K} \int_0^\infty \frac{1}{p} e^{-pz} J_0(pr) J_1(pa)\,dp$$

Soln: Let $V(r,z)$ is steady temperature at point (r,z), then it is governed by the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0$$

with boundary conditions

(i) $\frac{\partial V}{\partial z} = \frac{-Q}{2K} \qquad 0 \le r \le a, \ z=0$

(ii) $\frac{\partial V}{\partial z} = 0 \qquad r>a, \ z=0$

Taking zeroth order Hankel transform of the eqn (1), we get

$$H_0\left\{\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right\}+H_0\left\{\frac{\partial^2 V}{\partial z^2}\right\}=0$$

$$-p^2\bar{V}+\frac{d^2\bar{V}}{dz^2}=0 \quad \text{(by using result (II))} \quad \text{——— (2)}$$

Its soln is

$$\bar{V}=Ae^{pz}+Be^{-pz}.$$

Boundedness of $\bar{V} \Rightarrow A=0$, so that

$$\bar{V}=Be^{-pz} \quad \text{——— (3)}$$

$$\Rightarrow \frac{\partial\bar{V}}{\partial z}=-pBe^{-pz} \quad (\because B=B(p))$$

$$\Rightarrow \left(\frac{\partial\bar{V}}{\partial z}\right)_{z=0}=-pB$$

$$\Rightarrow B=-\frac{1}{p}\left(\frac{\partial\bar{V}}{\partial z}\right)_{z=0}=-\frac{1}{p}H_0\left\{\frac{\partial V}{\partial z}\right\}$$

$$=-\frac{1}{p}\int_0^\infty \frac{\partial V}{\partial z}\, r J_0(pr)\,dr$$

$$=-\frac{1}{p}\left[\int_0^a \frac{-Q}{2K}\, r J_0(pr)\,dr+\underbrace{\int_a^\infty 0\cdot r J_0(pr)\,dr}_{0}\right]$$

(by using given conditions)

$$=\frac{Q}{2pK}\int_0^a \frac{1}{p}\frac{d}{dr}\{rJ_1(pr)\}\,dr \quad \left[\because rJ_0(pr)=\frac{1}{p}\frac{d}{dr}\{rJ_1(pr)\}\right]$$

i.e. $$B=\frac{Qa}{2p^2K}J_1(ap)$$

using it in (3), we have $\bar{V}=\frac{Qa}{2p^2K}e^{-pz}J_1(ap)$

Applying Hankel inverse formula

$$V=\frac{Qa}{2K}\int_0^\infty \frac{1}{p}e^{-pz}J_1(ap)J_0(pr)\,dp$$

Hence proved.