Electromagnetic Vector Potential and Scalar Potential

→ A vector potential is a vector field whose curl is a given vector field. This is similar to a scalar potential, which is a scalar field whose gradient is a given vector field.

→ We know Maxwell's magnetic field equation

$$\text{div}\,\vec{B} = \vec{\nabla}\cdot\vec{B} = 0 \quad (1)$$ (i.e. field vector $\vec{B}$ is solenoidal)

We also know div of curl of any vector is always zero, so $\vec{B}$ can be expressed as the curl of a vector i.e.

$$\boxed{\vec{B} = \text{Curl}\,\vec{A} = \vec{\nabla}\times\vec{A}} \quad -(2)$$

Here $\vec{A}$ is known as electromagnetic vector potential.

→ We also know Maxwell's equation

$$\text{Curl}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad -(3)$$

Putting the value of $\vec{B}$ from eq (2) in eq (3)

$$\text{Curl}\,\vec{E} = -\frac{\partial}{\partial t}(\text{Curl}\,\vec{A})$$

$$\text{Curl}\left[\vec{E} + \frac{\partial\vec{A}}{\partial t}\right] = 0 \quad -(4)$$

This equation implies that the vector $\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right)$ has zero curl i.e. $\left[\vec{E} + \frac{\partial\vec{A}}{\partial t}\right]$ is irrotational] and we know that curl of gradient of any scalar function is always zero; hence the vector $\left[\vec{E} + \frac{\partial\vec{A}}{\partial t}\right]$ can be expressed as the gradient of a scalar function S

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = \text{grad}\,S \quad -(5)$$

or equivalently in conventional form

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\text{grad}\,\phi \quad \text{or} \quad E = -\left[\frac{\partial A}{\partial t} + \text{grad}\,\phi\right] \quad -(6)$$

Here ϕ is called scalar potential function.

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Electromagnetic wave equation for Vector and Scalar potential (wave equation satisfied by $\vec{E}$ and $\vec{B}$)

We know Maxwell's field equations are

$$\text{div } \vec{D} = \vec{\nabla}\cdot\vec{D} = \rho \quad -(1)$$

$$\text{div } \vec{B} = \vec{\nabla}\cdot\vec{B} = 0 \quad -(2)$$

$$\text{Curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad -(3)$$

$$\text{Curl } \vec{H} = \vec{J} + \frac{\partial D}{\partial t} \quad -(4)$$

Vector $\vec{B}$ can be expressed as the Curl of a vector

$$\vec{B} = \text{Curl } \vec{A} = \vec{\nabla}\times\vec{A} \quad -(5)$$ [$\vec{A}$ → electromagnetic vector Potential]

Substituting the value of $\vec{B}$ in eq (3)

$$\text{Curl } \vec{E} = -\frac{\partial}{\partial t}(\text{Curl } \vec{A}) \Rightarrow \text{Curl}\left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0$$

- We know that the Curl of gradient of any scalar function is always zero, so vector $\left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right]$ can be expressed as the gradient of Scalar function -

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = \text{grad } S = -\text{grad } \phi$$

or

$$\vec{E} = -\left[\frac{\partial A}{\partial t} + \text{grad } \phi\right] \quad -(6)$$ [ϕ → Scalar potential function]

eq (5) and (6) express electromagnetic field vector $\vec{E}$ and $\vec{B}$ in terms of electromagnetic scalar and vector Potential ϕ and $\vec{A}$

Now Maxwell eq (4) is

$$\text{Curl } \vec{H} = J + \frac{\partial \vec{D}}{\partial t}$$

in terms of $\vec{B}$ and $\vec{E}$ ($B = \mu H$ and $D = \epsilon E$)

$$\text{Curl } \vec{B} = \mu\vec{J} + \mu\epsilon_0\frac{\partial E}{\partial t} \quad -(7)$$

using eq (5) and (6) we get

$$\text{Curl Curl } \vec{A} = \mu\vec{J} + \mu\epsilon\frac{\partial}{\partial t}\left[-\frac{\partial \vec{A}}{\partial t} - \text{grad } \phi\right]$$

$$\text{grad div } A - \nabla^2 A = \mu\vec{J} - \mu\epsilon_0\frac{\partial^2 A}{\partial t^2} - \mu\epsilon\frac{\partial}{\partial t}\text{grad } \phi$$

Rearranging weight

$$\nabla^2 A - \mu \epsilon_0 \frac{\partial^2 A}{\partial t^2} - \text{grad}\left[\text{div} A + \mu \epsilon \frac{\partial \phi}{\partial t}\right] = -\mu \vec{J} \quad (8)$$

Similarly eq (1) in term of E takes the form $(D = \epsilon E)$

$$\text{div}\, E = \frac{\rho}{\epsilon}.$$

using (5) weight

$$\text{div}\left[-\frac{\partial A}{\partial t} - \text{grad}\, \phi\right] = \frac{\rho}{\epsilon_0}$$

$$\text{div grad}\, \phi + \frac{\partial (\text{div} A)}{\partial t} = -\frac{\rho}{\epsilon_0}$$

Adding and subtracting $\mu \epsilon \frac{\partial^2 \phi}{\partial t^2}$ and rearranging weight.

$$\nabla^2 \phi - \mu \epsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left[\text{div} \vec{A} + \mu \epsilon \frac{\partial \phi}{\partial t}\right] = \frac{-\rho}{\epsilon_0} \quad (9)$$

eq (8) and (9) represent two inhomogeneous wave equation in terms of electromagnetic Potentials $\vec{A}$ and ϕ in place of four Maxwell's equation. These equation hence $\vec{A}$ and ϕ Coupled with each other.

These inhomogeneous wave equation Can be uncoupled by using Concept that $\vec{A}$ and ϕ are arbitrary? $\vec{A}$ and ϕ are chosen subject to the Condition.

$$\text{div} A + \mu \epsilon \frac{\partial \phi}{\partial t} = 0 \quad (10)$$

Since div $\vec{A}$ is so far arbitrary we Can do this. This Condition Known as "Lorentz Condition",

Ten eq (8) and (9) for electromagnetic vector and Potential

$$\nabla^2 A - \mu \epsilon \frac{\partial^2 A}{\partial t^2} = -\mu J \quad (11)$$

$$\nabla^2 \phi - \mu \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon} \quad (12)$$

If $\vec{A}$ and ϕ Satisfied these equation then field determined by eq (5) and (6) Satisfy Maxwell's equation.

If Current density obey ohm's law $(\vec{J} = \sigma \vec{E})$ and the charge density is zero. [as we should have in the interior of a Conductor.

~~and effect the charge density is zero~~, In that case Lorentz condition (10) will be

$$\text{div}\,\vec{A} + \mu\epsilon_0 \frac{\partial\phi}{\partial t} + \mu\sigma\phi = 0 \quad \text{—} \quad (13)$$

and then eq (11) and (12)

$$\nabla^2 A - \mu\sigma \frac{\partial A}{\partial t} - \mu\epsilon_0 \frac{\partial^2 A}{\partial t^2} = 0 \quad \text{—} \quad (14)$$

$$\nabla^2 \phi - \mu\sigma \frac{\partial\phi}{\partial t} \; \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = 0 \quad (15)$$

If the source $\vec{J}$ and ρ are absent then eq (14) and (15) reduce to homogeneous equat

$$\nabla^2 \vec{A} - \mu\epsilon\, \partial^2 \vec{A} = 0 \quad \text{—} \quad 16$$

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2\phi}{\partial t^2} = 0 \quad \text{—} \quad (17)$$

This equation represent an extention of Poisson's equation, obtaining by adding the time derivatives. And are known as D'Alimbert equations. Introducing D'Alimbert's operator

$$\Box^2 = \nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2} \quad \text{—} \quad (18)$$

eq (16) and (17) take the similar to Poisson's equation

$$\Box^2 \vec{A} = 0 \quad \text{and} \quad \Box^2 \phi = 0 \quad \text{—} \quad (19)$$

in term of D'Alimbert operator eq (11) and 12 take the form

$$\Box^2 \vec{A} = -\mu\vec{J} \quad \text{and} \quad \Box^2 \phi = -\frac{\rho}{\epsilon_0} \quad \text{—} \quad (20)$$

for free space $\mu = \mu_0$ and $\epsilon = \epsilon_0$ then

$$\Box^2 \vec{A} = -\mu_0 \vec{J} \quad \text{and} \quad \Box^2 \phi = -\frac{\rho}{\epsilon_0} \quad \text{—} \quad (21)$$

Here $\Box^2 = \nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

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The retarded solution for scalar and vector potentials may now be written as –

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{[\rho(r')]dv'}{R(r,r')} \quad \text{--- (a)}$$

$$A(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{[J(r')]dv'}{R(r,r')} \quad \text{--- (b)}$$

(1)

Here $[\rho(r')]$ and $[J(r')]$ within square brackets represent, value of charge and current at the earlier time $t' = t - \frac{R}{c}$, i

⇒ Electromagnetic potentials $\vec{A}$ and ϕ, the electromagnetic field vectors, in general calculated by the relation.

$$\vec{B} = \text{Curl}\,\vec{A} \text{ and } \vec{E} = -\text{grad}\,\phi - \frac{\partial\vec{A}}{\partial t} \quad (2)$$

Lienard - Wiechert Potentials

Lienard - Wichert potentials are scalar and vector potentials produced by a moving point charge. For obtaining this potential, consider a moving volume carrying with it a fixed charge distribution [For example a uniformly charged spherical volume, moving through space along a prescribed trajectory]. The field due to a point charge may then be obtained by taking properly the limit of the field due to such a charge distribution.

The scalar and vector potentials due to moving charge and current distributions at point $\vec{r}$ and time t are given by the retarded potentials.

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right) dV'}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(r', t-\frac{R}{c}\right)}{R(r,r')} dV'$$

$$A(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} dV'$$

$$= \frac{\mu_0}{4\pi}\int \frac{J\left(r', t-\frac{R}{c}\right)}{R(\vec{r},\vec{r}')} dV'$$

Here $R(\vec{r},\vec{r}') = |\vec{r}-\vec{r}'|$ — (1)

→ The volume in which ρ and $\vec{J}$ are different from zero, is not specified. To remove the difficulty let us introduce one dimensional Dirac delta function having properties

(i) $\delta(x) = 0$ for $x \neq 0$ and $\delta(0) \neq 0$

(ii) $\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$

(iii) $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$

(iv) $\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a)$

— (2)

where $f(x)$ is regular, continuous and differentiable functions of x.

with δ function the scalar and vector potentials given by (1) may be written in the form.

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{V'}\int_{-\infty}^{+\infty} \frac{\rho(\vec{r}',t')\,\delta\left[t'-\left(t-\frac{R}{c}\right)\right]}{R(r,r')} dV'\,dt'$$

$$A(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{V'}\int_{-\infty}^{+\infty} \frac{J(\vec{r}',t')}{R(\vec{r},\vec{r}')}\,\delta\left[\left\{t'-\left(t-\frac{R}{c}\right)\right\}\right] dV'\,dt'$$ — (3)

Now $\rho(r',t')dv' = dq$ is the charge in the volume element dv' and similarly $\vec{J}(\vec{r},t') = \rho(r',t')\vec{v}(t')$ is the current density produced by an element of charge density $\rho(\vec{r}',t')$ moving with velocity $\vec{v}(t')$.
This gives -

$$\vec{J}(r',t')dv' = \rho(r',t')v(t')dV' = \vec{v}(t')\,dq \quad (4)$$

So we get

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\int_{-\infty}^{+\infty} \frac{dq\,\delta(t'-t+\frac{R}{c})\,dt'}{R(\vec{r},\vec{r}')}$$

$$A(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{V'}\int_{-\infty}^{+\infty} \frac{\vec{v}(t')\,dq\,\delta(t'-t+\frac{R}{c})\,dt'}{R(r,r')} \quad (\delta)$$

in the charge distribution limit to point charge q, we can immediately write eq (5) as.

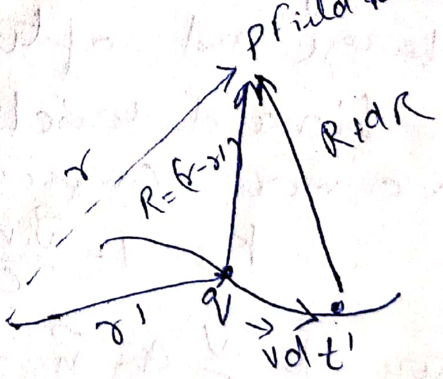
$$\phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} \frac{\delta(t'-t+R/c)\,dt'}{R(r,r')}$$

$$A(\vec{r},t) = \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty} \frac{v(t')\,\delta(t'-t+\frac{R}{c})\,dt'}{R(r,r')} \quad - (\gamma)$$

These integrals can be evaluated if we choose instead of t' a new fixed time t_1 given by -

$$t_1 = t' - t + \frac{R}{c} \quad (7a)$$

we get $$\frac{dt_1}{dt'} = 1 + \frac{1}{c}\frac{dR}{dt'} \quad (7b)$$



we have $R^2 = \vec{R}\cdot\vec{R}$

or $$2R\frac{dR}{dt'} = \vec{R}\cdot\frac{d\vec{R}}{dt'} + \frac{d\vec{R}}{dt'}\cdot\vec{R}$$

$$\frac{dR}{dt'} = \frac{\vec{R}}{R}\cdot\frac{d\vec{R}}{dt'} \quad - (8)$$

From fig we have $\vec{R} = (\vec{R}+d\vec{R}) + \vec{v}dt'$ i.e $\frac{d\vec{R}}{dt'} = -\vec{v}$

Using eq (8) $$\frac{dR}{dt'} = -\vec{v}\cdot\frac{\vec{R}}{R} \quad - (9)$$

Putting this value of $\frac{dR}{dt}$ in eq (7b) we get

$$\frac{dt'}{dr'} = 1 - \frac{\vec{v}}{c}\cdot\frac{\vec{R}}{R} = \frac{R-(\vec{v}\cdot\vec{R}/c)}{R} \text{ or } dt' = \frac{R\,dt_1}{R-(\frac{\vec{v}}{c})\cdot\vec{R}} \quad - (10)$$

7 (9) and (10) in (6) we get

$$\left.\begin{aligned}\phi(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{\delta(t_1)\,dt_1}{R-\left\{\frac{\vec{v}\cdot\vec{R}}{c}\right\}}\\ \vec{A}(\vec{r},t) &= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\frac{\vec{v}(t')\,\delta(t_1)\,dt_1}{R-\left\{\frac{\vec{v}\cdot\vec{R}}{c}\right\}}\end{aligned}\right| \quad (11)$$

Now use property of δ function, we have

$$\left.\begin{aligned}\phi(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R-\frac{\vec{v}\cdot\vec{R}}{c}}\right]_{\text{at } t_1=0}\\ \vec{A}(\vec{r},t) &= \frac{\mu_0 q}{4\pi}\left[\frac{\vec{v}}{R-\frac{\vec{v}\cdot\vec{R}}{c}}\right]_{\text{at } t_1=0}\end{aligned}\right| \quad (12)$$

But $t_1=0$ implies that $t'-t+\frac{R}{c}=0$ or $t'=t-\frac{R}{c}$, the retarded time

So eq (12) may be expressed as

$$\left.\begin{aligned}\phi(r,t) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R-\frac{\vec{v}\cdot\vec{R}}{c}}\right]_{\text{retarded}}\\ \vec{A}(\vec{r},t) &= \frac{\mu_0 q}{4\pi}\left[\frac{v}{R-\frac{\vec{v}\cdot\vec{R}}{c}}\right]_{\text{retarded}}\end{aligned}\right| \quad (13)$$

These potentials are velocity dependent and were first given by A. Lienard and E. Wiechert for moving point charge and after their name they are known as Lienard-Wiechert potentials. This may be expressed in a more compact form if we substitute

$$K = 1-\frac{\vec{v}\cdot n}{c} = 1-\vec{\beta}\cdot\vec{n} \quad (14)$$

Where $\vec{\beta}=\frac{\vec{v}}{c}$ and $\vec{n}=\frac{\vec{R}}{R}$ is unit vector directed from source point q to the field point P. then eq (13) will be

$$\left.\begin{aligned}\phi(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{KR}\right]_{\text{retarded}}\\ A(\vec{r},t) &= \frac{\mu_0 q}{4\pi}\left[\frac{\vec{\beta}c}{KR}\right]_{\text{retarded}}\end{aligned}\right| \quad (15)$$

Here $\vec{\beta}=\frac{\vec{v}}{c}$