The Electromagnetic fields of a uniformly moving point charge

The electric and magnetic field of a point charge in motion are given as –

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{(\vec{R}-R\vec{\beta})(1-\beta^2)}{(R-\vec{R}\cdot\vec{\beta})^3} + \frac{\{\vec{R}\times(\vec{R}-R\vec{\beta})\times\dot{\vec{\beta}}\}}{c(R-\vec{R}\cdot\vec{\beta})^3}\right]_{\text{retarded}} \quad (1)$$

$$\vec{B}(\vec{r},t) = \frac{\hat{n}\times\vec{E}}{c} = \frac{\mu_0 c q}{4\pi}\left[\frac{(\vec{\beta}\times\hat{n})(1-\beta^2)}{(R-\vec{R}\cdot\vec{\beta})^3} + \frac{\hat{n}\times\{\vec{R}\times(\vec{R}-R\vec{\beta})\times\dot{\vec{\beta}}\}}{cR(R-\vec{R}\cdot\vec{\beta})^3}\right]_{\text{retarded}} \quad (2)$$

For uniformly moving point charge, $\vec{v} = c\beta = \text{Constant}$, this emplies that
Therefore for a uniformly moving point charge eq (1) takes the form. $\dot{\beta} = 0$.

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{(\vec{R}-R\vec{\beta})(1-\beta^2)}{(R-\vec{R}\cdot\vec{\beta})^3}\right]_{\text{retarded}} \quad ---- \quad (3)$$

Now as the particle is moving with constant velocity, we have

$$[\vec{\beta}]_{\text{retarted}} = \vec{\beta}$$

and hence $[1-\beta^2]_{\text{retarded}} = 1-\beta^2$ $\quad ---- \quad (4)$

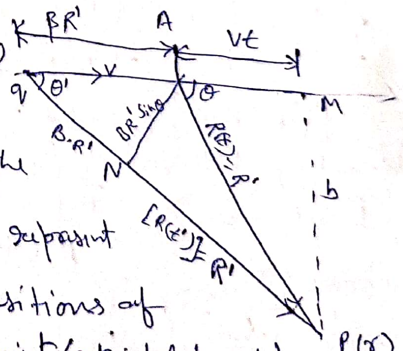
Hence equation (3) may be expressed as –

$$\vec{E}(\vec{r},t) = \frac{q(1-\beta^2)}{4\pi\epsilon_0}\left[\frac{\vec{R}-R\vec{\beta}}{(R-\vec{R}\cdot\vec{\beta})^3}\right]_{\text{retarted}} \quad (5)$$

Magnetic field may be computed by the relation –

$$\vec{B}(\vec{r},t) = \left[\frac{\hat{n}\times\vec{E}(\vec{r},t)}{c}\right]_{\text{retarded}} \quad --- \quad (6)$$

Equation (5) and (6) represent electromagnetic field in term of retarded position of the point charge.



Now for a uniformly moving point charge, it is desirable to represent these fields in terms of the present position of the charge.

In fig A and A' occupied by the charge represent respectively the retarded and present positions of the charge with reference to (P) the field point (or point of observation).
- Retarded distance of field point P(r) from source point A'(r') is $[R(t')] = R'$ and the present distance of field point P from those the source point A is $R(t) = R$. Electromagnetic signal starting from the source point A' reaches the field point P in time R'/c. of the charge itself moving with velocity v reaches the point A,

(8c)

The present position therefore the distance $A'A = vR'/c = \beta R'$, θ' is angle between v and R' and θ is the angle between v and R, AN is the perpendicular drawn from present position on $A'P$ and PM is the perpendicular drawn from the present position on $A'P$ and PM is the perpendicular drawn from field point P on the direction of velocity (V) of the moving charge –

from fig – $[\vec{R} - R\vec{\beta}]_{\text{retarded}} = \vec{R}' - R'\vec{\beta} = \vec{R}$ [from $\Delta AA'P$] (7)

and $\quad b = R'\sin\theta' = R\sin\theta$

This implies that $\quad \beta R'\sin\theta' = \beta R\sin\theta$

so
$$\beta^2 R'^2 \sin^2\theta' = \beta^2 R^2 \sin^2\theta$$
$$\beta^2 R'^2 (1-\cos^2\theta') = \beta^2 R^2 (1-\cos^2\theta)$$
$$\beta^2 R'^2 - (\beta R'\cos\theta')^2 = \beta^2 R^2 - (\beta R\cos\theta)^2$$
$$\beta^2 R'^2 - (\vec{\beta}\cdot\vec{R}')^2 = \beta^2 R^2 - (\vec{\beta}\cdot\vec{R})^2 \quad \text{— (8)}$$

Squaring eq (7) we get
$$R'^2 - 2R'(\vec{\beta}\cdot\vec{R}') + R'^2\beta^2 = R^2 \quad \text{— (9)}$$

Substracting eq (8) from (9) we get
$$R'^2 - 2R'(\vec{\beta}\cdot\vec{R}') + (\vec{\beta}\cdot\vec{R}')^2 = R^2 - \beta^2 R^2 + (\vec{\beta}\cdot\vec{R})^2 \quad \text{— (10)}$$

Now $[\{R - \vec{R}\cdot\vec{\beta}\}^2]_{\text{retarded}} = (R' - \vec{R}'\cdot\vec{\beta})^2 = R'^2 - 2R'(\vec{R}'\cdot\vec{\beta}) + (\vec{R}'\cdot\vec{\beta})^2$

$= R'^2 - 2R'(\vec{\beta}\cdot\vec{R}') + (\vec{\beta}\cdot\vec{R}')^2 = R^2 - \beta^2 R^2 + (\vec{\beta}\cdot\vec{R})^2$

(using (10))

$= R^2 - \beta^2 R^2 + (\vec{\beta}\cdot\vec{R})^2$

$= R^2 - \beta^2 R^2 + \beta^2 R^2\cos^2\theta = R^2 - \beta^2 R^2(1-\cos^2\theta)$

$$[\{R - \vec{R}\cdot\vec{\beta}\}^2]_{\text{retarded}} = R^2(1 - \beta^2\sin^2\theta) \quad \text{— (11)}$$

Substituting ~~the~~ values from (7) and (11) in eq (5) we get
$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\vec{R}(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{3/2}} \quad \text{— (12)}$$

Now from (2) using $\dot{\vec{\beta}} = 0$ we have
$$\vec{B}(r,t) = \left[\frac{\hat{n}\times\vec{E}}{c}\right]_{\text{retarded}} = \frac{\mu_0 c q}{4\pi}\left[\frac{(\vec{\beta}\times\vec{R})(1-\beta^2)}{(R - \vec{R}\cdot\vec{\beta})^3}\right]_{\text{retarded}}$$
$$= \frac{\mu_0 c q}{4\pi}(1-\beta^2)\frac{[\vec{\beta}\times\vec{R}]_{\text{retarded}}}{[(R - \vec{R}\cdot\vec{\beta})_{\text{retarded}}]^3} \quad \text{— (13)}$$

Now from fig —

$$PM = R'\sin\theta' = R\sin\theta$$

This gives

$$\beta R'\sin\theta' = \beta R\sin\theta$$

$$[\vec{\beta}\times\vec{R}'] = \vec{\beta}\times\vec{R}$$

$$[\vec{\beta}\times R]_{\text{retarded}} = \vec{\beta}\times\vec{R} \quad — (14)$$

using (11) and (14) we get

$$\vec{B}(r,t) = \frac{\mu_0 c q}{4\pi}\frac{(\vec{\beta}\times\vec{R})(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}} \quad — (15)$$

Eq (12) and (15) represent desired values of EMF vector $\vec{E}$ and $\vec{B}$ for a uniformly moving charge.

Discussion - from eq (12) and (15) we come to the following conclusion —

(1) Static case → for smaller velocities ($\beta\to 0$), the em field vector $\vec{E}$ and $\vec{B}$ take the form

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q\vec{R}}{R^3} \quad — (16)$$

which is Coulomb's law. This shows that for non-relativistic velocities the field is spherical symmetric; and

$$\vec{B}(\vec{r},t) = \frac{\mu_0 c q}{4\pi}\frac{\vec{\beta}\times\vec{R}}{R^3} = \frac{\mu_0 q}{4\pi}\frac{\vec{v}\times\vec{R}}{R^3} \quad \left[\text{Since } \beta = \frac{v}{c}\right]$$

$$= \frac{\mu_0 I dt}{4\pi}\frac{\vec{v}\times\vec{R}}{R^3} = \frac{\mu_0}{4\pi}\frac{I(vdt)\times\vec{R}}{R^3}$$

$$= \frac{\mu_0}{4\pi}\frac{I d\vec{l}\times\vec{R}}{R^3} \quad — (17)$$

which is Biot - Savart field.

(2) for high velocities - $\beta\to 1$ or $v\to c$, both field magnitudes depend on the angles θ between the direction of motion of charge and radius vector $\vec{R}$. From eq (12) electric field is greater for greater values of θ, that means field is not spherically symmetric, but it is stronger in the direction perpendicular to velocity. If $E_\perp$ and $E_\parallel$ are electric fields perpendicular and parallel to velocity then

$$E_{\perp}(\theta = 90^\circ) = \frac{1}{4\pi\epsilon_0} \frac{q\vec{R}}{R^3\sqrt{(1-\beta^2)}}$$

$$E_{\parallel}(\theta = 0^\circ) = \frac{1}{4\pi\epsilon_0} \frac{q\vec{R}(1-\beta^2)}{R^3}$$

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This shows that electric field is increased in a direction perpendicular to direction of motion in the ratio of $\frac{1}{\sqrt{(1-\beta^2)}}$ while in the direction of motion the field is decreased in the ratio $(1-\beta^2)$. there so indicating that the electric field of a moving charge is $\{[1/(1-\beta^2)^{3/2}]$ Times Stronger in the direction perpendicular to its motion than the field at the same distance along the direction of motion.

(3) Poynting vector $\vec{S} = \vec{E} \times \vec{H} = \frac{\vec{E} \times \vec{B}}{\mu_0}$

$$= \frac{\vec{E} \times (\vec{V} \times \vec{E})}{\mu_0 C^2} = \frac{(\vec{E}\cdot\vec{E})\vec{v} - (\vec{E}\cdot\vec{v})E}{\mu_0 C^2}$$

when v approaches c, we have seen that electric field is perpendicular to $\vec{v}$ and so $\vec{E}\cdot\vec{V} = 0$

then $\boxed{\vec{S} = \frac{E^2 \vec{V}}{\mu_0 C^2} = \epsilon_0 E^2 \vec{v}}$

Since $E \propto \frac{1}{R^2}$ we note that

$$\boxed{S \propto \frac{1}{R^4}}$$

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In order to calculate the energy radiated by the charged particle, we must integrate the normal Component of $\vec{S}$ over the Surface of sphere. If we take the radius of sphere to be R, then the energy radiated will be proportional to $\frac{1}{R^2}$ Since the element of Surface area involves R^2, therefore as $R \to \infty$, the energy radiated tends to zero. So we Conclude that the charged particles moving with constant velocity can not radiate energy