Finite Hankel Transform

The finite Hankel transform of the function $f(x)$, $0 \le x \le a$ of order n is denoted by $H_n\{f(x)\}$ and is defined by

$$H_n\{f(x)\} = \bar{f}_n(p_i) = \int_0^a x f(x) J_n(p_i x)\, dx \quad \text{--- (1)}$$

where p_i is the positive root of the equation

$$J_n(a p_i) = 0$$

Inversion Formula

The inverse finite Hankel transform $f(x)$ of $\bar{f}_n(p_i)$ is given by

$$f(x) = H_n^{-1}\{\bar{f}_n(p_i)\} = \frac{2}{a^2} \sum_{i=1}^{\infty} \bar{f}_n(p_i) \frac{J_n(p_i x)}{J_{n+1}^2(a p_i)} \quad \text{--- (2)}$$

where p_i $(0 < p_1 < p_2 < \cdots)$ are the roots of the eqn. $J_n(a p_i) = 0$

Since $J_{n+1}(a p_i) = -J_n'(a p_i)$

So (2) can be written as

$$f(x) = H_n^{-1}\{\bar{f}_n(p_i)\} = \frac{2}{a^2} \sum_{i=1}^{\infty} \bar{f}_n(p_i) \frac{J_n(a p_i)}{J_n'^2(a p_i)} \quad \text{--- (3)}$$

In particular, the zero th order finite Hankel transform and its inversion formula are then defined by

$$H_0\{f(x)\} = \bar{f}_0(p_i) = \int_0^a x f(x) J_0(p_i x)\, dx \quad \text{--- (4)}$$

and

$$H_0^{-1}\{\bar{f}_0(p_i)\} = f(x) = \frac{2}{a^2} \sum_p \bar{f}_0(p_i) \frac{J_0(a p_i)}{J_1^2(a p_i)} \quad \text{--- (5)}$$

where summation is taken over all positive roots of $J_0(ap) = 0$

Examples

(Ex. 1) Find the finite Hankel transform of

$$f(x) = x^n, \quad 0 \le x \le a$$

Soln. we know that

$$H_n\{f(x)\} = \int_0^a x f(x) J_n(px)\,dx$$

$$\therefore H_n\{x^n\} = \int_0^a x \cdot x^n J_n(px)\,dx$$

$$= \int_0^a x^{n+1} J_n(px)\,dx \quad \text{--- (1)}$$

Also we have

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

$$\Rightarrow \frac{d}{dx}\left(x^{n+1} J_{n+1}(px)\right) = p\, x^{n+1} J_n(px)$$

using it in (1) we get

$$H_n\{x^n\} = \int_0^a \frac{1}{p} \frac{d}{dx}\left(x^{n+1} J_{n+1}(px)\right) dx$$

$$= \frac{1}{p} \left| x^{n+1} J_{n+1}(px) \right|_0^a$$

$$\therefore \boxed{H_n\{x^n\} = \frac{a^{n+1}}{p} J_{n+1}(ap)}$$

In particular $(n=1)$

$$H_1\{x\} = \frac{a^2}{p} J_2(ap)$$

and for $n = 0$

$$H_0\{1\} = \frac{a^2}{p} J_1(ap)$$

(Ex. 2) Prove that finite Hankel transform of order n of

$$f(x) = \frac{2^{1+n-m}}{\Gamma m-n} x^n (1-x^2)^{m-n-1}$$

$$0 < x < 1$$

is $p^{n-m} J_m(p)$

Soln: We have

$$H_n\{f(x)\} = \int_0^1 x\, f(x)\, J_n(px)\, dx$$

$$= \int_0^1 \frac{2^{1+n-m}}{\Gamma(m-n)}\, x^{n+1}(1-x^2)^{m-n-1} J_n(px)\, dx$$

$$= \int_0^1 \frac{2^{1+n-m}}{\Gamma(m-n)}\, x^{n+1}(1-x^2)^{m-n-1} \sum_{r=0}^{\infty} \frac{(-1)^r (px)^{n+2r}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}$$

by def. of $J_n(x)$

$$= \frac{2}{\Gamma(m-n)} \sum_{r=0}^{\infty} \frac{(-1)^r p^{n+2r}}{2^{m+2r}\, r!\, \Gamma(n+r+1)} \int_0^1 x^{2n+2r+1}(1-x^2)^{m-n-1} dx$$

put $x^2 = t \Rightarrow x = t^{1/2}$
then $dx = \frac{1}{2} t^{-1/2} dt$

$$= \frac{2}{\Gamma(m-n)} \sum_{r=0}^{\infty} \frac{(-1)^r p^{n+2r}}{2^{m+2r}\, r!\, \Gamma(n+r+1)} \cdot \frac{1}{2}\int_0^1 t^{n+r+1-1}(1-t)^{m-n-1} dt$$

$$= \frac{1}{\Gamma(m-n)} \sum_{r=0}^{\infty} \frac{(-1)^r p^{n+2r}}{2^{m+2r}\, r!\, \Gamma(n+r+1)} \frac{\Gamma(n+r+1)\,\Gamma(m-n)}{\Gamma(n+r+1+m-n)}$$

$$= p^{n-m} \sum_{r=0}^{\infty} \frac{(-1)^r p^{m+2r}}{2^{m+2r}\, r!\, \Gamma(m+r+1)}$$

$$= p^{n-m} J_m(p)$$

Finite Hankel transform of Derivative

For $0 \le x \le 1$, Hankel transform of $\frac{d}{dx} f(x)$ or $f'(x)$ of order n is given by

$$H_n\left\{\frac{df}{dx}\right\} = \int_0^1 x \frac{df}{dx} J_n(px)\, dx$$

$$= \left| x f(x) J_n(px) \right|_0^1 - \int_0^1 f(x) \frac{d}{dx}\{x J_n(px)\} dx$$

~~Assuming $x f(x) \to 0$ as $x \to 0$~~

$$= -\int_0^1 f(x) \frac{d}{dx}\{x J_n(px)\} dx \quad (1)$$

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by using recurrence relations for $J_n(x)$ and replacing x by px, we can obtain

$$\frac{d}{dx}\{x J_n(px)\} = \frac{px}{2n}\left[(n+1)J_{n-1}(px) - (n-1)J_{n+1}(px)\right]$$

using it in ① we get

$$H_n\left\{\frac{df}{dx}\right\} = -\int_0^1 f(x)\left[\frac{px}{2n}\{(n+1)J_{n-1}(px) - (n-1)J_{n+1}(px)\}\right]dx$$

$$= \frac{p}{2n}\left[(n-1)\int_0^1 x f(x) J_{n+1}(px)\,dx - (n+1)\int x f(x) J_{n-1}(px)\,dx\right]$$

$$= \frac{p}{2n}\left[(n-1) H_{n+1}\{f(x)\} - (n+1) H_{n-1}\{f(x)\}\right]$$

In Particular, for $n=1$

$$\boxed{H_1\left\{\frac{df}{dx}\right\} = -p\, H_0\{f(x)\}}$$

For $n=2$

$$H_2\left\{\frac{df}{dx}\right\} = \frac{p}{4}\left[H_3\{f(x)\} - 3H_1\{f(x)\}\right]$$

→ Finite Hankel Transform of $\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f$ in $0 \le x \le a$ where p is a root of $J_n(ap) = 0$

$$H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f\right\} = -ap\, f(a) J_n'(ap) - p^2 H_n\{f(x)\}$$

Deduction: If $a = 1$ then

$$H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f(x)\right\} = -p\, f(1) J_n'(p) - p^2 H_n\{f(x)\}$$