Hankel Transform

The infinite Hankel transform of $f(x)$, $0 < x < \infty$ is defined as

$$H\{f(x)\} = \bar{F}(P) = \int_0^\infty f(x) \cdot x J_n(Px)\, dx \quad \text{--- (I)}$$

where $J_n(Px)$ is the Bessel function of first kind of order n. Also here $\bar{F}(P)$ is defined as Hankel transform of order n of $f(x)$ so it is also denoted by $H_n\{f(x)\}$ or $\bar{F}_n(P)$.

Inversion formula

If $\bar{f}_n(P)$ is the Hankel transform of order n of the function $f(x)$, then

$$H^{-1}\{\bar{f}_n(P)\} = f(x) = \int_0^\infty \bar{f}_n(P)\, P J_n(Px)\, dP \quad \text{--- (2)}$$

NOTE - Since the Bessel function $J_n(Px)$ is involved in the kernel of the Hankel transform so to solve the problems of the Hankel transform easily, it is necessary a brief study of Bessel function.

Some Important results on Bessel function

(I) Bessel function $J_n(x)$ of first kind of order n is defined as

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Also $J_n(-x) = (-1)^n J_n(x)$

& $J_{-n}(x) = (-1)^n J_n(x)$

(II) Reccurrence Relations for $J_n(x)$

(i) $\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$

(ii) $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$

(iii) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(iv) $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

(v) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(vi) $2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$.

(III) Infinite Integrals involving Bessel Functions

(i) $\int_0^\infty e^{-ax} J_0(px)\, dx = \frac{1}{\sqrt{a^2+p^2}}$

(ii) $\int_0^\infty e^{-ax} J_1(px)\, dx = \frac{1}{p} - \frac{a}{p\sqrt{a^2+p^2}}$

(iii) $\int_0^\infty x e^{-ax} J_0(px)\, dx = \frac{a}{(a^2+p^2)^{3/2}}$

(iv) $\int_0^\infty x e^{-ax} J_1(px)\, dx = \frac{p}{(a^2+p^2)^{3/2}}$.

(v) $\int_0^\infty \frac{e^{-ax}}{x} J_1(px)\, dx = \frac{\sqrt{a^2+p^2} - a}{p}$

(vi) $\int_0^\infty J_n(px)\, dx = \frac{1}{p}$ $(n = 0, 1, 2 \ldots)$

(vii) $\int_0^\infty \sin ax\, J_0(px)\, dx = \begin{cases} \frac{1}{\sqrt{a^2-p^2}} & a > p \\ 0 & a < p \end{cases}$

(viii) $\int_0^\infty \cos ax\, J_0(px)\, dx = \begin{cases} \frac{1}{\sqrt{p^2-a^2}} & p > a \\ 0 & p < a \end{cases}$

Properties of Hankel transform

1 Linear property.

$$H\{c_1 f_1(x) \pm c_2 g(x) \pm \cdots\}$$

$$= c_1 H\{f(x)\} \pm c_2 H\{g(x)\} \pm \cdots$$

where c_1 & c_2 are constant

2 Change of scale property

If $H\{f(x)\} = \bar{F}(p)$ then $H\{f(ax)\} = \frac{1}{a^2}\bar{F}(p/a)$

3 Hankel transform of Derivatives

If $H\{f(x)\} = \bar{f}_n(p)$ then

$$H\{f'(x)\} = \frac{-p}{2n}\left[(n+1)\bar{f}_{n-1}(p) - (n-1)\bar{f}_{n+1}(p)\right] \quad -③$$

In particular (i) when $n=1$

$$H_1\{f'(x)\} = -p\bar{f}_0(p) = \bar{f}_1(p)$$

(ii) $n=2$

$$H_2\{f'(x)\} = \bar{f}_2(p) = \frac{-p}{4}\left[3\bar{f}_1(p) - \bar{f}_3(p)\right]$$

(iii) when $n=3$

$$H_3\{f'(x)\} = \bar{f}_3(p) = \frac{-p}{6}\left[4\bar{f}_2(p) - 2\bar{f}_4(p)\right]$$

From ③ we can also obtain

$$H_n\{f''(x)\} = \bar{F}_n''(p) = \frac{p^2}{4}\left[\left(\frac{n+1}{n-1}\right)\bar{f}_{n-2}(p) - 2\left(\frac{n^2-3}{n^2-1}\right)\bar{f}_n(p) + \left(\frac{n-1}{n+1}\right)\bar{f}_{n+2}(p)\right]$$

Parseval's Theorem

If $\bar{F}(p)$ and $\bar{g}(p)$ be the Hankel transforms of the functions $f(x)$ and $g(x)$ respectively then
$$\int_0^\infty x\, f(x)\, g(x)\, dx = \int_0^\infty p\, \bar{f}(p)\, \bar{g}(p)\, dp$$

Proof. we know that
$$\bar{f}(p) = \int_0^\infty f(x) \cdot x J_n(px)\, dx$$
$$\& \quad \bar{g}(p) = \int_0^\infty g(x) \cdot x J_n(px)\, dx$$
we have
$$\int_0^\infty p \bar{f}(p) \bar{g}(p)\, dp = \int_0^\infty p \bar{f}(p) \left(\int_0^\infty g(x) \cdot x J_n(px)\, dx \right) dp$$
By changing the order of integration on RHS we get
$$\int_0^\infty p \bar{f}(p) \bar{g}(p)\, dp = \int_0^\infty x\, g(x)\, dx \cdot \int_0^\infty p \bar{f}(p) J_n(px)\, dp$$
$$= \int_0^\infty x\, g(x)\, dx \cdot f(x)$$
$\because$ (by inversion formula)
$$= \int_0^\infty x\, f(x)\, g(x)\, dx$$

A Special Result

$$H_n \left\{ \frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} - \frac{n^2}{x^2} f \right\} = -p^2 \bar{f}_n(p)$$

(i) when $n=0$
$$H \left\{ \frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} \right\} = -p^2 \bar{f}(p)$$

(ii) when $n=1$
$$H \left\{ \frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} - \frac{1}{x^2} f \right\} = -p^2 \bar{f}_1(p)$$