Inverse Laplace Transform

If $L\{F(t)\} = f(P)$ then inverse laplace transform of $f(P)$ is $F(t)$ i.e.

$$L^{-1}\{f(P)\} = F(t)$$

Basic Formula

$$L^{-1}\{\tfrac{1}{p}\} = 1$$

$$L^{-1}\{\tfrac{1}{(P+a)}\} = e^{-at}$$

$$L^{-1}\{\tfrac{1}{P-a}\} = e^{at}$$

$$L^{-1}\{\tfrac{1}{p^n}\} = \frac{t^{n-1}}{\Gamma n}$$

$$L^{-1}\{\tfrac{1}{p^2+a^2}\} = \tfrac{1}{a}\sin at$$

$$L^{-1}\{\tfrac{p}{p^2+a^2}\} = \cos at$$

$$L^{-1}\{\tfrac{1}{p^2-a^2}\} = \tfrac{1}{a}\sinh at$$

$$L^{-1}\{\tfrac{p}{p^2-a^2}\} = \cosh at$$

$$L^{-1}\{e^{-aP}\} = \delta(t-a)$$

$$L^{-1}\{\tfrac{e^{-aP}}{P}\} = u(t-a)$$

$$L^{-1}\{\tfrac{1}{\sqrt{P^2+a^2}}\} = J_0(at)$$

$$L^{-1}\{\tfrac{1}{P\sqrt{P+1}}\} = \operatorname{erf}(\sqrt{t})$$

$$L^{-1}\{1\} = \delta(t)$$

UNIQUENESS OF INVERSE LAPLACE TRANSFORM

Lerch's Theorem:- Let $L\{F(t)\} = f(P)$. Let $F(t)$ be piecewise continuous in every finite interval $0 \le t \le a$ and is of exponential order for $t > a$, then inverse laplace transform $F(t)$ of $f(P)$ is unique.

Properties of Inverse Laplace transform

① Linear property

$$L^{-1}\{c_1 f_1(p) \pm c_2 f_2(p) \pm \cdots\}$$
$$= c_1 L^{-1}\{f_1(p)\} \pm c_2 L^{-1}\{f_2(p)\} \pm \cdots$$

e.g. Find $L^{-1}\left\{\frac{1}{p^3} - \frac{3}{p-2} + \frac{4}{p^2+1}\right\}$

we have

$$L^{-1}\left\{\frac{1}{p^3} - \frac{3}{p-2} + \frac{4}{p^2+1}\right\} = L^{-1}\left\{\frac{1}{p^3}\right\} - 3L^{-1}\left\{\frac{1}{p-2}\right\} + 4L^{-1}\left\{\frac{1}{p^2+1}\right\}$$
$$= \frac{t^2}{2} - 3e^{2t} + 4\sin t$$

② first shifting property.

$$L^{-1}\{f(p-a)\} = e^{at} L^{-1}\{f(p)\}$$
and $$L^{-1}\{f(p+a)\} = e^{-at} L^{-1}\{f(p)\}$$

e.g. Find $L^{-1}\left\{\frac{p+2}{(p+1)^2+9}\right\}$

we have $L^{-1}\left\{\frac{p+2}{(p+1)^2+9}\right\} = e^{-t} L^{-1}\left\{\frac{p+1}{p^2+9}\right\}$

by first shifting property

$$= e^{-t}\left[L^{-1}\left\{\frac{p}{p^2+9}\right\} + L^{-1}\left\{\frac{1}{p^2+9}\right\}\right]$$
$$= e^{-t}\left[\cos 3t + \frac{1}{3}\sin 3t\right]$$

Remark. — If we can find inverse laplace transform of $f(p)$ for p but there is $(p+a)$ or $(p-a)$ in place of p then we can use first shifting property.

(3) Second Shifting Theorem

If $L^{-1}\{f(p)\} = F(t)$ then

$$L^{-1}\{e^{-ap} f(p)\} = F(t-a)\, u(t-a)$$

where $u(t-a)$ is unit step function

e.g. Find $L^{-1}\{e^{-\pi p}/(p^2+1)\}$

Take $f(p) = \frac{1}{p^2+1}$ then $F(t) = L^{-1}\{f(p)\}$

$$= L^{-1}\{\tfrac{1}{p^2+1}\}$$

$$= \sin t$$

Now by second shifting theorem

$$L^{-1}\left\{\frac{e^{-\pi p}}{p^2+1}\right\} = \sin(t-\pi)\, u(t-\pi)$$

$$= -\sin t\, u(t-\pi)$$

(4) Change of scale property

If $L^{-1}\{f(p)\} = F(t)$ then

$$L^{-1}\{f(ap)\} = \frac{1}{a} F(t/a), \quad a>0$$

e.g. Find $L^{-1}\left\{\frac{p}{2p^2+8}\right\}$

we know that $L^{-1}\left\{\frac{p}{p^2+16}\right\} = \cos 4t$

then by change of scale property

$$L^{-1}\left\{\frac{2p}{(2p)^2+16}\right\} = \frac{1}{2}\cos\left(\frac{4t}{2}\right)$$

or $L^{-1}\left\{\frac{p}{2p^2+8}\right\} = \frac{1}{2}\cos 2t$

(5) Inverse Laplace transform of Derivative

$$L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n L^{-1}\{f(p)\}$$

For $n=1$, $\boxed{L^{-1}\{f'(p)\} = -t L^{-1}\{f(p)\}}$

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Remark:- It is very useful formula since it can be used from both sides i.e.

(i) when the given function (whose inverse Laplace transform to be found) can be written as derivative of a function, say $f(P)$ and inverse L.T. of $f(P)$ can be determined easily. e.g.-

$$L^{-1}\left\{\frac{2P}{(P^2+1)^2}\right\}$$

we know that $\frac{2P}{(P^2+1)^2} = \frac{d}{dP}\frac{1}{(P^2+1)}$

Hence $f(P) = \frac{1}{P^2+1}$ & $L^{-1}(f(P)) = \sin t$

Now by formula.

$$L^{-1}\left\{\frac{d}{dP}f(P)\right\} = -t\,L^{-1}\{f(P)\}$$

we have

$$L^{-1}\left\{\frac{2P}{(P^2+1)^2}\right\} = L^{-1}\left\{\frac{d}{dP}\frac{1}{(P^2+1)}\right\} = -t\,L^{-1}\left\{\frac{1}{P^2+1}\right\}$$

$$= -t\sin t$$

(ii) Some times to determine the inverse Laplace transform of the given function is complicated but inverse transform of its derivative can be calculated easily. then we use the above formula as

$$L^{-1}\{f(P)\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{dP}f(P)\right\}$$

e.g. find $L^{-1}\{\tan^{-1}P\}$

Here the inverse transform of the derivative of $\tan^{-1}P$ i.e. $\frac{1}{1+P^2}$ is easier than that of $\tan^{-1}P$, so from the above formula we have

$$L^{-1}\{\tan^{-1}p\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{dp}\tan^{-1}p\right\}$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= -\frac{1}{t}\sin t.$$

6 Multiplication by p

If $L^{-1}\{f(p)\} = F(t)$ and $F(0)=0$

then $L^{-1}\{pf(p)\} = \frac{d}{dt}F(t)$ — (1)

It can be generalised as follow:

If $L^{-1}\{f(p)\} = F(t)$ and $F(0)=F'(0)=\cdots\cdots=F^{(n-1)}(0)=0$

then $L^{-1}\{p^n f(p)\} = \frac{d^n}{dt^n}F(t)$

e.g.

$$L^{-1}\left\{\frac{2p^2}{(p^2+1)^2}\right\} = L^{-1}\left\{p\cdot\frac{2p}{(p^2+1)^2}\right\}$$

$= L^{-1}\{pf(p)\}$ where $f(p) = \frac{2p}{p^2+1}$

& $F(t) = -t\sin t$

also $F(0)=0$

Now by (1)

$$L^{-1}\left\{\frac{2p^2}{(p^2+1)^2}\right\} = \frac{d}{dt}(-t\sin t)$$

$$= -(t\cos t + \sin t)$$

7 Division by p

If $L^{-1}\{f(p)\} = F(t)$ then

$$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(x)\,dx$$

Find $L^{-1}\left\{\frac{1}{p(p^2+9)}\right\}$

we take $\frac{1}{p^2+9} = f(p)$ then $F(t) = \frac{1}{3}\sin 3t$

Then $L^{-1}\left\{\frac{1}{p(p^2+9)}\right\} = L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t \frac{1}{3}\sin 3x\, dx$

$$= -\frac{1}{9}\left|\cos 3x\right|_0^t$$

$$= -\frac{1}{9}(\cos 3t - 1)$$

8 Convolution Theorem

The convolution of two functions $F(t)$ and $G(t)$ is denoted by $F*G$ and is defined as

$$F*G = \int_0^t F(u)\,G(t-u)\,du$$

$$= \int_0^t F(t-u)\,G(u)\,du$$

Theorem If $L^{-1}\{f(p)\} = F(t)$ & $L^{-1}\{g(p)\} = G(t)$ then

$$L^{-1}\{f(p)\cdot g(p)\} = F*G$$

$$= \int_0^t F(u)\,G(t-u)\,du$$

Remark— It is generally used to find inverse Laplace transform of product of two or more functions.