Laplace transforms of some Special functions

1 Laplace transform of Heaviside's unit step function →

Heaviside unit step function is defined as

$$H(t-a) = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \end{cases} \quad \text{--- (1)}$$

It is also denoted by $u(t-a)$ and its laplace transform

$$L\{H(t-a)\} = \int_0^\infty e^{-pt} H(t-a)\, dt$$

$$= \int_0^a e^{-pt} \cdot 0 \, dt + \int_a^\infty e^{-pt} \cdot 1 \, dt \quad \text{by (1)}$$

$$= 0 + \frac{e^{-ap}}{p}$$

$$\boxed{\therefore \ L\{H(t-a)\} = \frac{e^{-ap}}{p}}$$

2 Dirac delta function

$$\delta_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & 0 < t \leq \epsilon \\ 0 & t > \epsilon \end{cases}$$

and its laplace transform

$$L\{\delta_\epsilon(t)\} = \int_0^\infty e^{-pt} \delta_\epsilon(t)\, dt$$

$$= \int_0^\epsilon e^{-pt} \cdot \frac{1}{\epsilon}\, dt + \int_\epsilon^\infty e^{-pt} \cdot 0 \, dt$$

$$= \frac{1}{\epsilon} \int_0^\epsilon e^{-pt}\, dt + 0$$

$$\therefore L\{\delta_\epsilon(t)\} = \frac{1}{p\epsilon}(1 - e^{-p\epsilon})$$

In general,

$$\delta(t-a) = \begin{cases} \frac{1}{\epsilon} & a \le t \le a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

with properties.

(i) $\int_0^\infty \delta(t-a)\,dt = 1$

(ii) $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

where $f(t)$ is a function continuous at $t=a$

In particular when $f(t) = e^{-pt}$, we have

$$\int_0^\infty e^{-pt}\,\delta(t-a)\,dt = e^{-ap}$$

$$\Rightarrow \boxed{L\{\delta(t-a)\} = e^{-ap}}$$

and $\boxed{L\{\delta(t)\} = 1}$

③ <u>Bessel Functions</u>

Bessel function $J_n(x)$ of first kind and of order n

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Laplace Transform of $J_0(t)$ and $J_1(t)$

$$L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}}$$

and $$L\{J_1(t)\} = 1 - \frac{p}{\sqrt{p^2+1}}$$

(4) Error Function

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$$

and

$$\boxed{L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{p\sqrt{p+1}}}$$

Ex. 1 Find the Laplace transforms of the functions

(i) $\sin^2 3t$ (ii) $e^{-2t}\sin 4t$ (iii) $\cosh at \sin at$

Soln. (i) $\because \sin^2 3t = \frac{1-\cos 6t}{2}$

$$\therefore L\{\sin^2 3t\} = L\left\{\frac{1-\cos 6t}{2}\right\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 6t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+36}\right]$$

(ii) we have $L\{\sin 4t\} = \frac{4}{p^2+16}$

then by first shifting Theorem

$$L\{e^{-2t}\sin 4t\} = \frac{4}{(p+2)^2+16} = \frac{4}{p^2+4p+20}$$

(iii) $$L\{\cosh at \sin at\} = L\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\sin at\} + L\{e^{-at}\sin at\}\right]$$

$$= \frac{1}{2}\left[\frac{a}{(p-a)^2+a^2} + \frac{a}{(p+a)^2+a^2}\right]$$

by using first shifting theorem