Q. Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

By Gauss Divergence Theorem, we have —

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_V \nabla\cdot\vec{F}\,dv$$

$$\nabla\cdot\vec{F} = 4z - 2y + y = 4z - y$$

$$RHS = \int_0^1\int_0^1\int_0^1 (4z - y)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left[2z^2 - yz\right]_0^1 dy\,dx$$

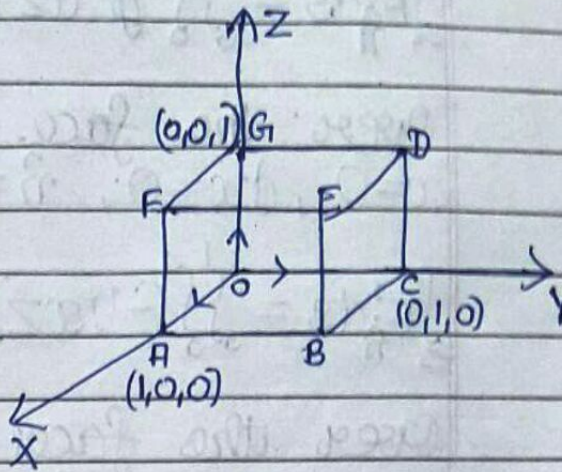
$$= \int_0^1\int_0^1 (2-y)\,dy\,dx$$

$$= \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right) dx$$

$$= \int_0^1 \frac{3}{2} dx$$

RHS $= \frac{3}{2}$




$\int \vec{F}\cdot\hat{n}\, ds =$ LHS

Over the face OABC $\longrightarrow$

$z=0,\ dz=0,\ \hat{n} = -\hat{k},\ ds = dx\,dy$

$$\int_{S_1} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (-yz)\, dx\, dy = 0 \quad \text{——} \ (1)$$

Over the face BCDE $\longrightarrow$

$y=1,\ dy=0\quad \hat{n} = \hat{j}\quad ds = dx\,dz$

$$\int_{S_2} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 (-y^2)\, dx\, dz$$

$$= -\int_0^1\int_0^1 dx\, dz = -1 \quad \text{——} \ (2)$$

Over the face DEFG $\longrightarrow$

$z=1,\ dz=0,\ \hat{n} = \hat{k}\quad ds = dx\,dy$

$$\int_{S_3} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 y\, dy\, dx$$

$$= \int_0^1 \frac{1}{2} dx = \frac{1}{2} \quad \text{——} \ (3)$$

Over the face AOGF $\longrightarrow$

$y=0, dy=0, \hat{n}=-\hat{j} \quad ds=dz\,dx$

$$\int_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 y^2\,dz\,dx = 0 \quad \text{--- (4)}$$

Over the face OCDG $\longrightarrow$

$x=0, dx=0, \hat{n}=-\hat{i}, ds=dy\,dz$

$$\int_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (-4xz)\,dy\,dz = 0 \quad \text{--- (5)}$$

Over the face ABEF $\longrightarrow$

$x=1, dx=0, \hat{n}=\hat{i}, ds=dy\,dz$

$$\int_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4xz\,dz\,dy$$

$$= \int_0^1\int_0^1 4z\,dz\,dy$$

$$= \int_0^1 2\,dy$$

$$= 2 \quad \text{--- (6)}$$

$$LHS = \int_S \vec{F}\cdot\hat{n}\,ds$$

$$= \int_{S_1} \vec{F}\cdot\hat{n}\,ds + \int_{S_2} \vec{F}\cdot\hat{n}\,ds + \int_{S_3} \vec{F}\cdot\hat{n}\,ds + \int_{S_4} \vec{F}\cdot\hat{n}\,ds + \int_{S_5} \vec{F}\cdot\hat{n}\,ds + \int_{S_6} \vec{F}\cdot\hat{n}\,ds$$

$$= 0 - 1 + \frac{1}{2} + 0 + 0 + 2$$

$LHS = 3/2$

$LHS = RHS$

Hence gauss divergence theorem is verified.