Stoke's Theorem:-

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

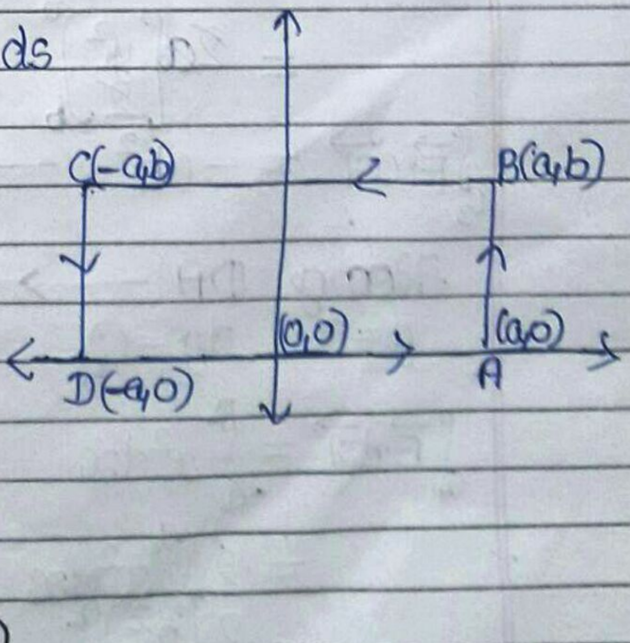
Cartesian Form of Stoke's Theorem:-

$$\int_C (F_1 dx + F_2 dy + F_3 dz) = \iint_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy\,dz + \iint_S \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dx\,dz$$
$$+ \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Q. Verify Stoke's Theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\,\hat{j}$ taken round the rectangle bounded by the line $x = \pm a$, $y = 0$, $y = b$.

By Stoke's Theorem, we have —

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$



Along AB $\longrightarrow$

$x = a$, $dx = 0$

$\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^b -2ay\,dy$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = -2a\left[\frac{y^2}{2}\right]_0^b$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = -ab^2 \quad \text{——— (1)}$$

Along BC $\longrightarrow$

$y=b\,,\; dy=0$

$$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{a}^{-a}(x^2+b^2)\,dx$$

$$= -2\int_{0}^{a}(x^2+b^2)\,dx$$

$$= -2\left[\frac{x^3}{3}+b^2x\right]_0^a$$

$$\int_{C_2}\vec{F}\cdot d\vec{r} = -2\left(\frac{a^3}{3}+ab^2\right) \quad \text{—— (2)}$$

Along CD $\longrightarrow$

$x=-a\,,\; dx=0$

$$\int_{C_3}\vec{F}\cdot d\vec{r} = \int_{b}^{0}-2(-a)y\,dy$$

$$= 2a\left[\frac{y^2}{2}\right]_b^0$$

$$\int_{C_3}\vec{F}\cdot d\vec{r} = -ab^2 \quad \text{—— (3)}$$

Along DA $\longrightarrow$

$y=0\,,\; dy=0$

$$\int_{C_4}\vec{F}\cdot d\vec{r} = \int_{-a}^{a}x^2\,dx$$

$$= 2\int_{0}^{a}x^2\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^a$$

$$\int_{C_4}\vec{F}\cdot d\vec{r} = \frac{2a^3}{3} \quad \text{—— (4)}$$

On adding ①, ② ③ & ④, we get →

$$LHS = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F} d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F} d\vec{r}$$

$$= -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$= -4ab^2$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{k}(-2y-2y)$$

$$\text{curl } \vec{F} = -4y\hat{k}$$

$$\hat{n} = \hat{k}$$

$$RHS = \int_S \text{curl } \vec{F}\cdot \hat{n}\, ds$$

$$= \int_0^b \int_{-a}^{a} (-4y)\, dx\, dy$$

$$= \int_0^b -4y\, [x]_{-a}^{a}\, dy$$

$$= \int_0^b -8ay\, dy$$

$$= -\cancel{8}^{4} a \left[\frac{y^2}{\cancel{2}}\right]_0^b$$

$$RHS = -4ab^2$$

$$LHS = RHS$$

Hence, Stoke's Theorem is verified.