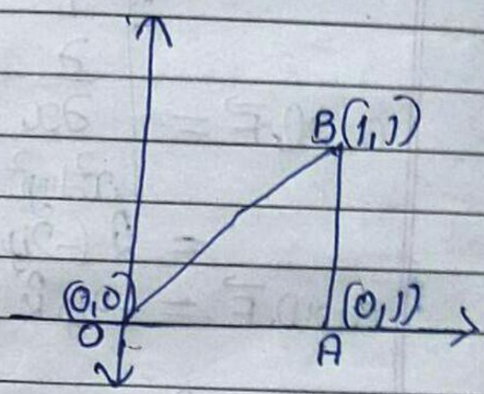
Q. Evaluate $\int \vec{F}\cdot d\vec{r}$ by Stoke's Theorem where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at (0,0,0) (1,0,0) and (1,1,0).

By Stoke's Theorem, we have —

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$



$$\text{curl}\,\vec{F} = +\hat{j} + (2x-2y)\hat{k}$$

$$\hat{n} = \hat{k}$$

$$\int_S \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^x 2(x-y)\,dy\,dx$$

$$= \int_0^1 -2\left[\frac{y^2}{2}\right]_0^x dx$$

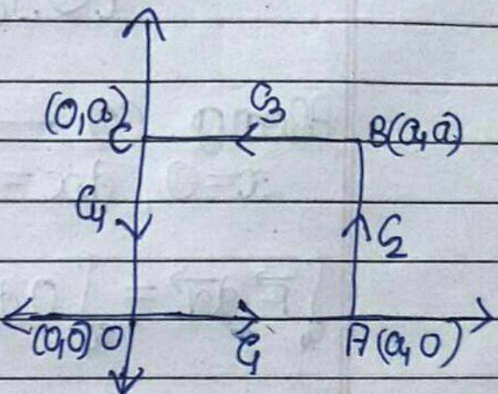
$$= \int_0^1 -x^2\,dx$$

$$\int_S \vec{F}\cdot d\vec{r} = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$ Ans

Q Verify Stoke's Theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square whose sides are $x=0$, $y=0$ $x=a$, $y=a$ in the plane $z=0$.

By Stoke's Theorem, we have —

$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$



Along OA ⟶

$y=0$, $dy=0$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \quad \text{—— (1)}$$

Along AB ⟶

$x=a$, $dx=0$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^a ay\,dy$$

$$= a\left[\frac{y^2}{2}\right]_0^a$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \frac{a^3}{2} \quad \text{—— (2)}$$

Along BC $\longrightarrow$

$y=a$, $dy=0$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_a^0 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3} \quad \text{---} \quad (3)$$

Along CO $\longrightarrow$

$x=0$, $dx=0$

$$\int_{C_4} \vec{F}\cdot d\vec{r} = \int_{C_4} 0\, dy = 0 \quad \text{---} \quad (4)$$

On adding eq (1), (2) (3) & (4), we get $\longrightarrow$

$$LHS = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}$$

$$= \cancel{\frac{a^3}{3}} + \frac{a^3}{2} - \cancel{\frac{a^3}{3}} + 0$$

$$= \frac{a^3}{2}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$\text{curl } \vec{F} = y\hat{k}$$

$$\hat{n} = \hat{k}$$

RHS

$$= \int_S \operatorname{curl} \vec{F} \cdot \hat{n}\, ds = \int_0^a \int_0^a y\, dy\, dx$$

$$= \int_0^a \frac{a^2}{2}\, dx$$

$$RHS = \frac{a^3}{2}$$

Since, LHS = RHS

Hence Stoke's Theorem is verified.