$\Rightarrow$ Green Theorem:- If C is a regular closed curve in the xy- plane and R be the region bounded by C, then

$$\int (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

where M & N are the functions of x and y.

Proof- By Stoke's theorem, we have

$$\int_C \vec{F}.\vec{dr} = \int_S (\text{Curl}\,\vec{F}.\hat{n}) ds$$

L.H.S $\int_C (M dx + N dy)$

$$d\vec{r} = \hat{i} dx + \hat{j} dy$$

let $\vec{F} = M\hat{i} + N\hat{j}$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$$

$$= -\hat{i}\frac{\partial N}{\partial z} + \hat{j}\left[-\frac{\partial M}{\partial z}\right] + \hat{k}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$\hat{n} = \hat{k}$ ($\because$ plane is x-y)

$\text{Curl}\,\vec{F}.\hat{k}$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

= R.H.S

Hence proved

Ques- By the use of Green theorem, show that area bounded by a simple closed curve C is given by
$\frac{1}{2}\int x\,dy - y\,dx$

By Green theorem,
We have

$$\int M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = -y$
and $N = x$

$\frac{\partial N}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$

$$= \iint_S [1 - (-1)]\,dx\,dy$$

$$= \iint_S 2\,dx\,dy$$

$$\int_C x\,dy - y\,dx = 2\iint_S dx\,dy$$

$$\frac{1}{2}\int_C x\,dy - y\,dx = 2A \qquad \because A = \iint_S dx\,dy$$

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

Ques- A vector field $\vec{F}$ is given by $F = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$. Evaluate the integral $\int_C \vec{F}\cdot\vec{dr}$ where C is the circular path given by $x^2+y^2=a^2$

$$\int_C \vec{F}\cdot\vec{dr} = \int_C \sin y\,dx + x(1+\cos y)\,dy$$

Here $N = x(1+\cos y)$
$= x + x\cos y$

and $M = \sin y$

$$\frac{\partial N}{\partial x} = 1 + \cos y$$

$$\frac{\partial M}{\partial y} = \cos y$$

$$= \iint_S (1 + \cos y - \cos y)\, dx\, dy$$

$$= \iint_S dx\,dy$$

$$= \iint_S dx\,dy$$

$$= A$$

$$= \pi a^2$$

<u>Ques</u>- Use Green theorem to evaluate $\int_C (x^2 + xy)\,dx + (x^2 + y^2)\,dy$ where C is the square formed by the lines $y = \pm 1, x = \pm 1$

<u>Solution</u> $\int (x^2 + xy)\,dx + (x^2 + y^2)\,dy$

$$= \iint_S (2x - \cancel{2}x)\, dx\, dy$$

$$= \iint_S x\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \int_{-1}^{1} \frac{x^2}{2}\, dy = 0$$

$$= \int_{-1}^{1}$$