

Envelope & Evolute

Envelope - The eqⁿ of envelope of the family of curve $f(x, y, \alpha) = 0$ where α is a parameter, is obtained by eliminating α between the equations

$$\begin{aligned} f(x, y, \alpha) &= 0 \\ \frac{\partial f(x, y, \alpha)}{\partial \alpha} &= 0 \end{aligned}$$

Q Find the eqⁿ of envelope of the curve $y = mx + \frac{1}{m}$

$$\frac{\partial y}{\partial m} = x - \frac{1}{m^2} = 0$$

$$x = \frac{1}{m^2}$$

$$m^2 = \frac{1}{x}$$

$$m = \frac{1}{\sqrt{x}}$$

$$y = \frac{1}{\sqrt{x}} x + \sqrt{x}$$

$$y = 2\sqrt{x}$$

$$\boxed{y^2 = 4x}$$

This is the required eqⁿ of envelope of the given family of curve.

Evolutes - The locus of centres of curvatures for a curve is called its evolute.

$(\bar{x}, \bar{y}) \rightarrow$ Centre of curvature, where,

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} \quad , \quad \bar{y} = y + \frac{1+y_1^2}{y_2}$$

Q Find the eqⁿ of evolute of the curve $y^2 = 4ax$.

$$2y y_1 = 4a$$

$$y_1 = \frac{2a}{y}$$

$$y_2 = -\frac{y}{2a} y_1$$

$$y_2 = -\frac{2a}{y^2} \times \frac{2a}{y} = -\frac{4a^2}{y^3}$$

Now, if (\bar{x}, \bar{y}) be the centre of curvature, then —

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\bar{x} = \frac{x - \frac{2a(1+4a^2)}{y^2}}{-\frac{4a^2}{y^3}}$$

$$\bar{x} = x + \frac{y^2 + 4a^2}{2a}$$

$$\bar{x} = \frac{2ax + y^2 + 4a^2}{2a} = \frac{6ax + 4a^2}{2a} = 3x + 2a \quad \text{①}$$

$$\begin{aligned}
 \bar{y} &= y + \frac{1+y^2}{y} \\
 &= y + \left(\frac{1+4a^2}{y^2} - \frac{4a^2}{y^3} \right) \\
 &= y - \frac{(y^2+4a^2)y}{4a^2} \\
 &= y \left(\frac{4a^2 - y^2 - 4a^2}{4a^2} \right) \\
 &= \frac{-y^3}{4a^2}
 \end{aligned}$$

$$\begin{aligned}
 \bar{y} &= \frac{-4ax}{4a^2} & \bar{y} &= \frac{-(4ax)^{3/2}}{4a^2} \\
 \bar{y} &= \frac{-x}{a} & \bar{y} &= \frac{-2x^{3/2}}{\sqrt{a}} \quad \text{--- (2)}
 \end{aligned}$$

from eq-1, we get \rightarrow

$$x = \frac{\bar{x} - 2a}{3}$$

$$\bar{y} = \frac{-2x^{3/2}}{\sqrt{a}}$$

$$\bar{y}^2 = \frac{4(\bar{x} - 2a)^3}{9a}$$

$$27\bar{y}^2 a = 4(\bar{x} - 2a)^3$$

Replace \bar{x} by x & \bar{y} by $y \rightarrow$

$$27ay^2 = 4(x - 2a)^3 \quad \underline{\text{Ans}}$$