Euler-Cauchy Equation
(or Homogeneous linear differential eq^n)

An equation of the form -

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-1} x \frac{dy}{dx} + a_n y = Q$$

where a_i's are constants and Q is the function of x, is called Homogeneous Linear differential equation.

Put $\quad z = \log x \quad$ or $\quad x = e^z$

Let $\quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$

$$\rightarrow x \frac{dy}{dx} = \frac{dy}{dz} = Dy$$

where $D = \frac{d}{dz}$

$$x \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y.$$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y \text{ -----}$$

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Q: solve -

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$$

Put $z = \log x$ or $x = e^z$

$$D(D-1)(D-2)y + 2D(D-1)y + 2y = 10\left(e^z + e^{-z}\right)$$

$$D(D-1)\left[D - \cancel{2} + \cancel{2}\right]y + 2y = 10\left(e^z + e^{-z}\right)$$

$$(D^2 - D)Dy + 2y = 10\left(e^z + e^{-z}\right)$$

$$D^3y - D^2y + 2y = 10\left(e^z + e^{-z}\right)$$

$$(D^3 - D^2 + 2)y = 10\left(e^z + e^{-z}\right)$$

A.E. $\Rightarrow m^3 - m^2 + 2 = 0$

$$\cancel{(m+1)(m^2 - m - m + 2) = 0}$$

$$\cancel{(m+1)(m-1)(m\quad) = 0}$$

$$(m+1)(m^2 - 2m + 2) = 0$$

$$m = \frac{+2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{2 \pm 2i}{2} = 1 \pm i$$

$m = -1$

~~C.F. = $C_1 e^{-x}$~~

~~$CF = C_1 e^{-z} + e^z\left[C_2 \cos x + C_3 \sin x\right]$~~

$$\underline{CF} = C_1 e^{-z} + e^z\left[C_2 \cos z + C_3 \sin z\right]$$

$$P.I. = \frac{1}{(D^3 - D^2 + 2)} \cdot 10(e^z + e^{-z})$$

$$= \frac{10e^z}{2} + \frac{10e^{-z} \cdot z}{(3D^2 - 2D)}$$

$$= \frac{10e^z}{2} + \frac{10ze^{-z}}{(3+2)}$$

$$= 5e^z + 2ze^{-z}$$

$$y = C.F. + P.I.$$

$$y = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z) + 5e^z + 2ze^{-z}$$

$$\cancel{y = c_1 x^{-1}}$$

$$y = c_1 x^{-1} + x(c_2 \cos(\log x) + c_3 \sin(\log x)) + 5x + 2\log x \, x^{-1}$$