Maxwell's Equation

The divergence ($\vec{\nabla}\cdot$ _) and Curl ($\vec{\nabla}\times$ —) relation of electromagnetic fields are called maxwell's equation. These equations bear the same relations to electromagnetism that Newton's laws of motion do to Mechanics.

In absence of any dielectric or magnetic material the four maxwell's equations are stated in the integral form as below-

1- $\oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$ (Gauss law in electrostatic)

2- $\oint \vec{B}\cdot d\vec{A} = 0$ (Gauss law in Magneto static)

3- $\oint \vec{E}\cdot d\vec{l} = -\frac{d\phi_B}{dt}$ (Faraday's law of electromagnetic Induction)

4- $\oint \vec{B}\cdot d\vec{l} = \mu_0\left(i+\epsilon_0\frac{d\phi_E}{dt}\right)$ (modified Ampere's law)

and in differential form

1. $\text{div}\,\vec{E} = \frac{\rho}{\epsilon_0}$ or $\text{div}\,\vec{D} = \rho$ or $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ or $\vec{\nabla}\cdot\vec{D} = \rho$

2. $\text{div}\,\vec{B} = 0$ or $\vec{\nabla}\cdot\vec{B} = 0$

3 - $\text{Curl}\,\vec{E} = -\frac{\partial B}{\partial t}$ or $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

4- $\text{Curl}\,\vec{B} = \mu_0\left[\vec{J}+\frac{\partial\vec{D}}{\partial t}\right]$ or $\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J}+\frac{\partial\vec{D}}{\partial t}\right]$

Unit - 03.

1- Maxwell's First equation.

(a) **Maxwell's equation in differential form**. Maxwell's first equation in differential form of Gauss law in electrostatic

$$\boxed{\operatorname{div} \vec{D} = \rho \quad \text{or} \quad \vec{\nabla}\cdot\vec{D} = \rho \quad \text{or} \quad \operatorname{div}\vec{E} = \frac{\rho}{\epsilon_0}} \quad \text{—(A)}$$

Derivation. According to Gauss's theorem for closed surface.

$$\int_S \vec{E}\cdot d\vec{S} = \frac{q}{\epsilon_0} \quad \text{—(1)}$$

If density of a uniform charge distribution at any point is ρ then -

$$q = \int_V \rho \, dV \quad (2)$$

from eq (1) and (2)

$$\int_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_V \rho\, dV \quad \text{-- (3)}$$

we know Gauss divergence theorem

$$\int_S \vec{E}\cdot d\vec{S} = \int_V (\vec{\nabla}\cdot\vec{E})\, dV \quad \text{—(4)}$$

So from (3) and (4)

$$\int_V (\vec{\nabla}\cdot\vec{E})\, dV = \frac{1}{\epsilon_0}\int_V \rho\, dV$$

$$\boxed{\text{or} \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{or} \quad \operatorname{div}\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{or} \quad \operatorname{div}\vec{D} = \rho} \quad (5)$$

(b) **Maxwell's first equation in Integral form** -

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{—(1)}$$ (Maxwell's equation in differential form)

Integrating this equation over volume we get

$$\int_V \vec{\nabla}\cdot\vec{D}\, dV = \int \rho\, dV = q \quad \text{—(2)}$$

But from Gauss divergence theorem

$$\int_V (\vec{\nabla}\cdot\vec{D})\, dV = \int_S \vec{D}\cdot d\vec{S} \quad \text{—(3)}$$ So from eq (2) and (3)

$$\boxed{\int_S \vec{D}\cdot d\vec{S} = \int \rho\, dV = q} \quad \text{or} \quad \boxed{\int_S \vec{E}\cdot d\vec{S} = \frac{q}{\epsilon_0}} \quad \text{— (4)}$$

Unit - 03

(3) Maxwell's Second equation.

in differential form. It is differential form of Gauss law in Magnetostatics.

$$\boxed{\text{div}\,\vec{B} = 0 \text{ or } \vec{\nabla}\cdot\vec{B} = 0} \quad -(B)$$

The net magnet flux through a closed surface is always zero

$$\int_S \vec{B}\cdot d\vec{S} = 0 \quad \ldots \quad (1)$$

we know Gauss divergence theorem

$$\int_S \vec{B}\cdot d\vec{S} = \int_V (\vec{\nabla}\cdot\vec{B})\,dV \quad -(2)$$

This equation valid for any arbitrary volume, so the integration must be zero. Hence from (1) and (2)

$$(\vec{\nabla}\cdot\vec{B}) = 0 \text{ or } \boxed{\text{div}\,\vec{B} = 0} \quad -(3)$$

(b) Integral form.

we know differential form $\vec{\nabla}\cdot\vec{B} = 0$ - (1)

Integrating this equation overall volume V then.

$$\int(\vec{\nabla}\cdot\vec{B})V = 0 \quad -(2)$$

we know Gauss divergence theorem.

$$\int_V (\vec{\nabla}\cdot\vec{B})\,dV = \int_S \vec{B}\cdot d\vec{S} \ldots (3)$$

So from (2) and (3)

$$\boxed{\int_S \vec{B}\cdot d\vec{S} = 0} \quad (4)$$

Unit-03

3. Maxwell IIIrd equation.

(a) In differential form – It is Faraday's law of electromagnetic Ind

$$\boxed{\text{Curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \quad -(c).$$

According to faraday's law, the Induced e.m.f. ($e = \oint \vec{E}\cdot d\vec{l}$) Produced in a closed circuit is equal to the negative of the rate of change of magnetic flux linked with that circuit

i.e $\quad e = -\frac{d\phi}{dt} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{s}$ — ① and $e = \oint \vec{E}\cdot d\vec{l}$ ②

from ① and ② $\quad \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{s}$ — 3

or $\oint \vec{E}\cdot d\vec{l} = -\int_s \frac{\partial B}{\partial t}\cdot ds$ — ③

according stokes's theorem. $\oint \vec{E}\cdot d\vec{l} = \int_s (\text{Curl } \vec{E}).ds$ — ④

so from ③ and ④ we get $\int_s (\text{Curl } \vec{E})\cdot d\vec{s} = -\int_s \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$ — ⑤

This equation valid for any arbitrary surface, hence both the Vectors in the integral must be equal at each point, so

$$\boxed{\text{Curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \quad - ⑥$$

(b) In Integral form. $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$ — ① Integrating over a surface bounded by a curve we get $\int_s (\vec{\nabla}\times\vec{E}).d\vec{s} = -\int_c \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} = -\frac{\partial}{\partial t}\int_s \vec{B}\cdot d\vec{s}$ — ②

we know Stokes theorem, $\int_s (\vec{\nabla}\times\vec{E}) d\vec{s} = \oint \vec{E}\cdot d\vec{l}$ — ③

So from eq ② and ③ $\quad \boxed{\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_c \vec{B}\cdot d\vec{s}}$ — ③

So electromotive force e.m.f ($e = \oint \vec{E}\cdot d\vec{l}$) around a closed path is equal to negative rate of change of magnetic ~~field~~ flux bounded by the path.

(15) Unit-03 (16)

Maxwell IV equation

Induction in differential form It is modified Ampere's law, which modified by Maxwell..

$$\text{Curl } \vec{B} = \mu_0 \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] \quad \text{--- (1)}$$

According to Ampere's law. $\oint \vec{B} \cdot d\vec{l} = \mu_0 I = \mu_0 \int_S \vec{J} \cdot d\vec{S}$ - 0

and from Stoke's theorem $\oint \vec{B} \cdot d\vec{l} = \int_S (\text{Curl } \vec{B}) \cdot dS$

Hence $\int_S (\text{Curl } \vec{B}) \cdot dS = \mu_0 \int_S \vec{J} \cdot d\vec{S}$

So that $\boxed{\text{Curl } \vec{B} = \mu_0 \vec{J}}$ — (2)

This equation valid for steady current for varying electric field, [from equation of Continuity

$$\text{div } \vec{J} + \frac{\partial \vec{\rho}}{\partial t} = 0$$

Taking div of eq (2) we get $\text{div Curl } \vec{B} = \mu_0 \text{div} \vec{J}$

But $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$ So $\text{div } \vec{J} = 0$

eq (1) obtain from Ampere's law is not inaccordance with the equation of Continuity, Hence it needs Corrections, According to Maxwell, eq (2) is incomplete for the definition of total current density. for this Maxwell suggested that we must add some vector $\vec{J}_d$ to it. then total current density must be Solenoid i.e.

$$\text{Curl } \vec{B} = \mu_0 (\vec{J} + \vec{J}_d) \quad \text{--- (3)}$$

$$\therefore \quad 0 = \text{div} (J + J_d)$$

$$\text{or} \quad \text{div} . \vec{J}_d = -\text{div} J = -\frac{\partial \rho}{\partial t}$$

But $\vec{\nabla} \cdot \vec{D} = \rho$ (from Ist equation) $\Rightarrow \text{div } \vec{J}_d = \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{D})$

where $\vec{D}$ is electric displacement vector, Hence $\frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{D}) = \nabla \cdot \frac{\partial \vec{D}}{\partial t} = \text{div} \left(\frac{\partial \vec{D}}{\partial t} \right)$

$\text{div } \vec{J}_d = \text{div} \frac{\partial D}{\partial t}$ So that $\vec{J}_d = \frac{\partial D}{\partial t}$ — (4)

Hence eq (3) will be $\boxed{\text{Curl } \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)}$ — (5)

or $D = \epsilon_0 E$ so

$$\boxed{\text{Curl } B = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)} \quad \text{— (6)}$$

(D) Integral form of Maxwell's IV th equation.

$$\text{Curl } \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \quad -(1)$$

Integrating over the surface S we get

$$\int_S (\text{Curl } \vec{B}) \cdot d\vec{s} = \mu_0 \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot ds \quad -(2)$$

Using Stokes theorem, $\int_S (\text{Curl } \vec{B}) \cdot d\vec{s} = \oint \vec{B} \cdot d\vec{l} \quad -(3)$

So from (2) and (3)

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot ds} \quad -(4)$$

The magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displace and through any surface bounded by the path.

Maxwell's equation for free space

In a free medium devoid (without) of free charge and current ($\vec{\rho} = 0$ and $\vec{J} = 0$) then

$$\boxed{\begin{array}{ll} \vec{\nabla} \cdot \vec{E} = 0 & \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = \frac{-\partial B}{\partial t} & \vec{\nabla} \times \vec{B} = \mu \epsilon_0 \frac{\partial E}{\partial t} \end{array}}$$