Subject Name: Mathematic II

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Topic: Euler's Theorem

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Euler’s Theorem on Homogeneous Function

If u= f (x,y) is a homogeneous functions of degree n is x and y.

Then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

If u is homogeneous function in x, y and z of degree n, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

Euler’s Theorem For Homogeneous Functions of two variables

Theorem If u= f (x,y) is a homogeneous functions of degree n of x and y.

Then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Proof Since *f* is a homogeneous function of x and y of degree n, we can write

$$f(x,y)=x^n F\left(\frac{y}{x}\right)$$

Then $\frac{\partial f}{\partial x}=x^n F'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)+n\,x^{n-1}F\left(\frac{y}{x}\right)$

$$x\,\frac{\partial f}{\partial x}=-x^{n-1}\,y\,F'\left(\frac{y}{x}\right)+n\,x^n\,F\left(\frac{y}{x}\right) \qquad (1)$$

Again $\frac{\partial f}{\partial y}=x^n F'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$

$$y\,\frac{\partial f}{\partial y}=x^{n-1}\,y\,F'\left(\frac{y}{x}\right) \qquad (2)$$

Adding 1 and 2 we have

$$x\ \frac{\partial f}{\partial x} + y\ \frac{\partial f}{\partial y} = n\,x^n\,F\left(\frac{y}{x}\right) = n\,f$$

Q. If $u = \log\left(\frac{x^3 + y^3}{x + y}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$$

Solution We have

$$u = \log\left(\frac{x^3 + y^3}{x + y}\right),$$

$$e^u = \left(\frac{x^3 + y^3}{x + y}\right) = f(x, y)$$

This f (x,y) is a homogeneous functions of degree 2, then by Euler theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2.f$$

$$x\frac{\partial}{\partial x}\left(e^u\right) + y\frac{\partial}{\partial y}\left(e^u\right) = 2.e^u$$

$$x\,e^u\,\frac{\partial u}{\partial x} + y\,e^u\,\frac{\partial u}{\partial y} = 2.\ e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$$

Q. If $u = \tan^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$$

Solution We have

$$u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right),$$

$$\tan u = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) = f(x,y)$$

This f (x,y) is a homogeneous functions of degree 1/2, then by Euler theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}.f$$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \frac{1}{2}\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$$

Q. If $u = \sin^{-1}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2}$, then show that

$$(i)\ \ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left(13+\tan^2 u\right)$$

Solution We have

$$u = \sin^{-1}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2},$$

$$\sin u = \left(\frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}} \right)^{1/2} = f(x, y)$$

This f (x,y) is a homogeneous functions of degree -1/12, then by Euler theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -\frac{1}{12}.f$$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = -\frac{1}{12}\sin u$$

$$x \cos u\frac{\partial u}{\partial x} + y \cos u\frac{\partial u}{\partial y} = -\frac{1}{12}\sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{12}\tan u \qquad (1)$$

(ii) Differentiating equation (1) partially w.r.to x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{12} \sec^2 u\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = -(\frac{1}{12} \sec^2 u + 1)\frac{\partial u}{\partial x} \qquad (2)$$

Again, differentiating equation (1) partially w.r.to y, we get

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = -\frac{1}{12} \sec^2 u\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = -(\frac{1}{12} \sec^2 u + 1)\frac{\partial u}{\partial y} \qquad (3)$$

Multiplying equation (2) by *x* and equation (3) by *y* and then adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -(\frac{1}{12} \sec^2 u + 1) \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -(\frac{1}{12} \sec^2 u + 1)\left(-\frac{1}{12}\tan u \right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{144}(\sec^2 u+12)\tan u=\frac{\tan u}{144}(13+\tan^2 u)$$

Referential Books:

1. H.K. Dass, “Advanced Engineering Mathematics”, S. Chand & Co., 9th Revised Ed
2. A.R. Vasishtha, “Differential Calculus”, Krishna Pra. Media (P) Ltd.
3. V. Kumar, “Differential Calculus”, Epsilon Publishing House