Module - 1

Subject : - Mathematics

Class & Year:- M.A./ M.Sc. –IIndYear

Topic :- FUNCTIONAL ANALYSIS

(BANACH –SPACE)

By

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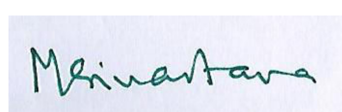

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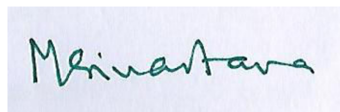

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FUNCTIONAL - ANALYSIS (BANACH –SPACE)

E-Content

M. A. /M.Sc. - FINAL

(2020 -2021)

LECTURE -1

Today we will start this topic with the definition of Normed linear space,Banach space and its theorems

Banach Space is a linear space which is also, in special way ,a complete metric space.

Normed Space : A normed linear space is a linear space N in which to each vector x there corresponds a real number, denoted by ||x|| and called the norm of x satisfies the properties

1) $||x|| > 0$

2) $||x|| = 0$ if and only if $x=0$

3) $||x + y|| \leq ||x|| + ||y||$

)4 $||\alpha x|| = |\alpha| \, ||x||$

The non negative real number $||x||$ is to be considered as the length of the vector x .If we consider $||x||$as a real function on N then this function is called the norm on N. The normed linear space Nis a metric space with respect to the ***metric d defined by*** $d(x, y) = ||x - y||$

Theorem 1: Let N be a normed space and x ,y ϵN ,then, $|\,||X|| - ||Y||\,| \leq || x-y ||$

Proof : we can write $||x|| = ||(x-y) + y || \leq ||x- y || +||y||$ by (2) giving

$||x|| - ||y|| \leq ||x- y]]$ **-----------(1)** and $||y|| = ||(y-x) +x || \leq ||y-x|| + ||x||$

gives$||y|| - ||x|| \leq ||y - x||= | - (x - y) ||=|-1 |\,||x-y||$ by (3) $\leq||x-y||$ **------- (2)**

(1) & (2) implies $|\,||X|| - ||Y||\,| \leq|| x-y ||$

From (1) we conclude that the norm is a continuous function

$x_n \to x$ this implies $|| x_n || \to ||x||$,this is clear from the fact that

$|\,|| x_n || - ||x||\,| \leq || x-y ||$,since $x_n \to x$ means that $|| x_n || - ||x|| \to 0$.In the same way we can prove that addition & multiplication are jointly continuous i.e.

$x_n \to x$ and $y_n \to y$ this implies $x_n + y_n \to x + y$ and $\alpha_n \to \alpha$ and $\alpha_n x_n \to \alpha x$ .These follow from$||(x_n + y_n) - (x + y) ||= ||(x_n - x) +(y_n - y) \leq|| x_n -x ||+|| y_n -y||$

and$|| \alpha_n x_n - \alpha x||= | \alpha_n| \, || x_n -x|| +| \alpha_n -\alpha| \, ||x||$

Theorem2 : Every convergent sequence in a normed linear space is a Cauchy sequence.

Proof: Assuming that a sequence $\{x_n\}$ in a normed linear space N converges to $x_0 \in N$, we claim that $\{x_n\}$ is a Cauchy sequence .

Given $\varepsilon > 0$, and the sequence $\{x_n\} \to x_0$,there exists a positive integer n_0 such that $n \geq n_0 \Rightarrow ||x_n - x_0|| < \varepsilon/2$, so that for all $m, n \geq n_0$, we have $||x_m - x_n|| =$ $||x_m - x_0 + x_0 - x_n|| \leq ||x_m - x_0|| + ||x_0 - x_n|| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, i.e. $||x_m - x_n|| < \varepsilon$ $\Rightarrow$ the sequence $\{x_n\}$ is a Cauchy sequence.

Theorem3 : the limit of a convergent sequence is unique.

Proof : Consider a sequence $\{x_n\}$ in a normed linear space N converges to two limits x, y such that x not equal to y i.e. $\{x_n\} \to x$ as well as $\{x_n\} \to y$ then

$||x_n - x|| \to 0$ and $||x_n - y| \to 0$,as $n \to \infty$ ----------------(1)

Now , $||x - y|| = ||x - x_n + x_n - y|| \leq ||x - x_n|| + ||x_n - y|| \leq |-1|||x_n - x|| + ||x_n - y|| \leq 0$ as $n \to \infty$, hence $||x - y|| = 0 \Rightarrow x=y$ therefore the limit of $\{x_n\}$ in N is unique

Lemma1: If a ,b ≥0 then $a^{1/p}b^{1/q} \leq a/p + b/q$ -------------------(1)

Proof :If a=0 ,b=0 then it is obvious ,let a>0,b>0.If $k \in (0,1)$,define f (t) for t≥1 by $f(t) = k(t-1) - t^k + 1$,then f(1)=0 and $f'(t) \geq 0$ and conclude that $t^k \leq kt + (1-k)$.If a≥b put t=a/b and k= 1/p ,if a<b ,put t=b/a and k= 1/q and in each case we get the required result.

Cor1. If we set $t=a^p b^{-q}$ in (1) we get $(a^p b^{-q})^{1/p} \leq 1/p\ a^p b^{-q}+1-1/p$ or $a\ b^{-q/p} \leq 1/p\ a^p b^{-q}$ $+1/q$ .Multiplying both sides by b^q ,this reduces to $a\ b \leq a^p/p + b^q/q$ --------------(2)

Theorem4: Holder' inequality in normed spaces:$\sum_{i=1}^{n} |x_i\ y_i| \leq \sum_{i=1}^{n} ||x||_p\ ||y||_q$.

where $x=(x_1,x_2\text{-------},x_n)$ & $y =(y_1,y_2\text{--------------},y_n)$ be n-tuples of scalars under the norm $||x|| = [\sum_{i=1}^{n} |x_i|^p]^{1/p}$

Proof:If x=0,y=0 then it is trivial .Assume $x\neq 0, y\neq 0$ put $a_i= |x_i|/||x||_p$ and ,$b_i= |y_i|/||y||_q$ and use the above lemma to obtain $|x_i||\ y_i|/\ ||x||_p||y||_q \leq a_i/p + b_i/q$. Add these inequality for i=1,2,------n,and conclude that $:\sum_{i=1}^{n} |x_i\ y_i| \leq$

$||\ x||_p||y||_q \leq 1/p+1/q =1$

Theorem5:Minkowski's**inequalityinnormed spaces :** $||x+y|| \leq ||x||_p + ||y||_q$.

The inequality is evident for p=1 ,so assume that p>1,use Holder's inequality to obtain

$||x+y||^p_p = \sum_{i=1}^{n} |x_i +y_i|^p = \sum_{i=1}^{n} |x_i +y_i||x_i +y_i|^{p-1} \leq \sum_{i=1}^{n} |x_i||x_i +y_i|^{p-1} + \sum_{i=1}^{n}$ $|y_i||x_i +y_i|^{p-1} \leq (||x||_p +||y||_q)||x+y||^{p/q}_p$ since $1/p +1/q =1$hence $p = (p-1) .q$

If p=q=2 then **Holder' inequality becomes Cauchy's inequality**

Theorem 6 : Show that the linear spacer R^n (Euclidean) & C^n of n-tuples $x=(x_1,x_2$-----,x_{n-1},x_n) of real &complex numbers are normed spaces under the norm $||x|| =\sum^n_{r=1} \{ (|x_i|^2)^{1/2}$,Also show that these spaces are complete and hence Banach

Proof:The normed linear space R^n& C^n are normed linear space,since

1) $||x||\geq 0$ 2) $||x||=0 \Leftrightarrow \sum^n_{r=1} \{ (|x_i|^2)^{1/2} \}=0 \Leftrightarrow \sum^n_{r=1} |x_i|^2 =0 \Leftrightarrow x_i=0$ for i=1,2,----n$\Leftrightarrow=(x_1,x_2$-----,-----,$x_{n-1},x_n)=0$ x =0

3) Taking $x= (x_1,x_2$-----,---$x_{n-1},x_n),y=(y_1,y_2$-----,---,$y_{n-1},y_n),= (x_1+ y_1$ --------$x_n+ y_n)$ we have $||x+y||^2= ||(x_1,x_2$-----,---$x_{n-1},x_n)+ (y_1,y_2$-----,-,$y_{n-1},y_n)||^2=|| (x_1+ y_1$ ---$x_n+ y_n)||$ $= \sum^n_{i=1} |x_i +y_i|^2 \leq \sum^n_{i=1} | x_i +y_i| (|x_i| +|y_i|\leq||x+y|| \ ||x|| + ||x+y|| \ ||y||\Rightarrow$ $||x+y||\leq ||x +y||$

4.$||\alpha x|| = \sum^n_{i=1} | \alpha x_i |^2 =|\alpha| \ ||x||$

Again to show that the normed spaces R^n & C^n are complete. Consider a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ of points in R^n & C^n so that x_i being n-tuples of real or complex numbers, we can write $x_1 =(x_{11}, x_{12}$------------ $x_{1n})$,$x_2= (x_{21}, x_{22}$------------ $x_{2n})$----------- $x_i = (x_{i1} x_{i2}$---------- $x_{in})$ .Now $\{x_i\}$ being a Cauchy sequence, for each $\varepsilon >0$ there exists n_0 such that for m, p $>n_0 \Rightarrow ||x_m - x_p||< \varepsilon \Rightarrow \sum^n_{i=1} \{|x_{mj} - x_{pj}|^2 \}^{1/2} < \varepsilon \Rightarrow \sum^n_{i=1}$ $\{|x_{mj} - x_{pj}|^2 \}< \varepsilon^2 \Rightarrow \{|x_{mj} - x_{pj}| \}^2 < \varepsilon$ for j=1,2,-------n$\Rightarrow$ each $\{ x_{j1}\}\infty_{j=1}$-------$\{ x_{jn}\}\infty_{j=1}$is a Cauchy sequence in real or complex points which are complete$\Rightarrow$

$\{ x_i\}\infty_{i=1-}$ converges to $z = (z_1,z_2$----------- $z_n)\in R^n$ or $C^n \Rightarrow R^n$ or C^n is a complete normed linear space

Theorem 7 : Let M be a closed linear subspace of a normed linear space N. If the norm of a coset x +M in the quotient space N/M is defined by

$||x+ M|| =\inf \{||x +m || : m\in M\}$ ---------- (1), then N/M is a normed linear space, Further ,if N is a Banach space then so is N.

Proof : We first prove that (1) defines norm in the required sense. It is obvious that **1)** $||x+ M|| \geq 0$ and since M is closed ,it is easy to see that 2) $||x+ M||= 0$ $\Leftrightarrow$ there exists a sequence $\{ m_k \}$ in M such that $|| x + m_k ||\to 0 \Leftrightarrow x$ is in M $\Leftrightarrow$ $x + M = M =$ the zero element of N/M. Next we have 3**)** $|| (x+ M) +(y+ M)|| =$ $||(x +y) +M || = \inf.\{||x+ y+ m|| : m\in M\} = \inf.\{||x+ y+ m +m'|| : m \,\&m'\in M\} =$ $\inf.\{||(x+ m) + (y+ m')|| : m\& m'\in M\} \leq \inf.\{||(x+ m) + (y+ m')|| : m \,\&\, m'\in M\} =$ $\inf.\{||(x+ m) : m\in M\} + \inf \{ ||(y+ m')|| : m'\in M\} =|| (x+ M) +(y+ M)||$, then we prove 4**)** $||\alpha(x + M) || = |\alpha| \, ||x +M||$ in the same manner.

Finally we assume that N is complete, and we show that N/M is also complete . Let$\{ x_n\}$ be a Cauchy sequence in N then it is sufficient to show that this sequence has a convergent subsequence. It is clearly possible to find a subsequence $\{x_n +M\}$ of the original Cauchy sequence such that

$||(x_1 + M) - (x_2 + M)|| < 1/2$, $||(x_2 + M) - (x_3 + M)|| < 1/4$,and in general,

$||(x_n + M) - (x_{n+1} + M)|| < 1/2^n$.We prove that this sequence is convergent in N/M. We choose a vector y_1 in $(x_1 + M)$ and y_2 in $(x_2 + M)$ such that $||y_1 - y_2|| < 1/2$, we next select a vector y_3 in $(x_3 + M)$ such that $||y_2 - y_3|| < 1/4$ continuing in this way we obtain a sequence $\{y_n\}$ in N such that

$||y_n - y_{n+1}|| < 1/2^n$.If $m < n$,then$||y_m - y_n|| = ||(y_m - y_{m+1})|| + ||(y_{m+1} - y_{m+2}||$ $+ \text{------------} ||y_{n-1} - y_n|| < 1/2^m + 1/2^{m+1} + \text{-----------} + 1/2^{n-1} < 1/2^{m-1}$, so $\{y_n\}$ is a Cauchy sequence. It follows from

$||(x_n + M) - (y + M) \leq ||y_n - y||$ that $x_n + M \rightarrow y + M$, so N/M is complete.

LECTURE -2

Now today we shall study operators & functionals in normed linear space and some of its Theorems.

Let N and N' be two normed spaces, then a one -one onto mapping $T : N \rightarrow N'$,is known as an operator or a transformation and the value of T at $x \in N$ is denoted by T (x) or T x.

The operator Tis known as a **linear operator transformation** if it satisfies the following two conditions:

$T(x + y) = T(x) + T(y)$ and $T(\alpha x) = \alpha T(x)$ i.e. T $\boldsymbol{(\alpha x + \beta y) = \alpha T(x) + \beta T(y)}$

T is bounded if $\boldsymbol{||T(x)|| \leq K\, ||x||}$

The operator T is continuous at a point $x_0 \in N$,if given $\varepsilon > 0$ there exists a $\delta(\varepsilon, x_0)$ such that $||T(x) - T(x_0)|| < \varepsilon$ whenever $||x - x_0|| < \delta$, T is continuous at every point of N , it is uniformly continuous if $\delta(x_0)$ is indepent of x_0 .

The norm of a bounded operator Tis defined as:

$$||T|| = \text{Sup}\{||T(x)|| \div ||x|| : x \neq 0\} \text{ ----------- } (1)$$

or equivalently $||T|| = \text{Sup}\{||T(x)|| : ||x|| \leq 1\}$ ---------------- (2)

and $|T|| = \text{Sup}\{||T(x)|| : ||x|| = 1\}$,if $N \neq 0$ ------- -(3)

we can express it as $|T|| = \text{Inf.}\{K : K \geq 0$ and $||T(x)|| \leq K\, ||x||$ for all x } ---------(4)

If N’ = R (normed space of reals) then T is known as a functional and denoted by f.

A normed linear space consisting of all bounded linear functionals over N is known as a **conjugate space denoted by** Ñ or N^{*}.

The set **R (T) = {T (x) ϵ N’: x ϵN} is known as the range space of the operator T** and the set **N(T) = {x ϵ N : T (x) =0 } is known as null space of T.**

Two operators T_1 &T_2 are equal if $T_1(x) = T_2(x)$, ∀ x

T is zero or **null operator if T (x) =0,** for every x

Tis known as identity operator and denoted by I , if **T (x) = x**, for every x

All continuous (bounded) linear transformations of N into N’ are denoted by

B(N,N’) where B stands for bounded .If N’ = R or C then B(N,R) or B(N,C) constitutes the **conjugate space** and **elements of N are called continuous linear functional or simply functionals.**

Theorem 8 : If T be a linear transformation from a normed space N into the normed space N’ ,then the following statements are equivalent:

1) T is continuous 2) T is continuous at origin i.e. $x_n \to x$ then $T(x_n) \to T(x)$

3) T is bounded i.e. there exists real k ≥0 such that $||T(x)|| \leq K ||x||$,for all x

4) The image T (S) of closed unit sphere S = {x : $||x|| < 1$} under Tis bounded subset of N'

Proof : (1) $\Leftrightarrow$ (2) .If T is continuous ,then since T(0) =0 it is certainly continuous at the origin . On the other hand ,if T is continuous at the origin at the origin then , $x_n \to x \Leftrightarrow x_n - x \to 0 \Rightarrow T(x_n - x) \to 0 \Leftrightarrow T(x_n) - T(x) \to 0 \Leftrightarrow T(x_n) \to T(x)$, so T is continuous.

$\Leftrightarrow$(3) .It is obvious that (3) $\Rightarrow$ (2) ,for if such a K exists, then $x_n \to 0$, clearly implies that $T(x_n) \to 0$. To show that (2) $\Rightarrow$ (3) ,we assume that there is no such K, it follows from this that for each positive integer n we can find a vector x_n such that $||T(x_n)|| > n||x_n|| \Rightarrow ||T(x_n) \div n||x_n|| > 1$.If we put $y_n = x_n / n||x_n||$,then it is easy to see that $y_n \to 0$,but $T(y_n)$ does not converges to 0,so T is not continuous at the origin.

(2)$\Leftrightarrow$(4). Since a non empty subset of a normed linear space is bounded $\Leftrightarrow$ it is contained in a closed sphere centered on the origin ,it is evident that (3)$\Rightarrow$ (4);for if $||x|| \leq 1$, then $|| T(x)|| \leq K$. To show that (4) $\Rightarrow$ (3) ,we assume that T(S) is contained in a closed sphere of radius K centered on the origin. If $x \div= 0$,then $T(x) = 0$ and clearly $|| T(x)|| \leq K. ||x||$,and if $x \neq 0$ then $x \div ||x|| \in S$ and therefore $T(x \div ||x||) \leq K$ so again we have $||T(x)|| \leq K||x||$.

Theorem9: If Tis a linear transformation of normed space N into normed space N' then the inverse of T i.e. T^{-1} exists and is continuous on its domain of definition iff there exists a constant $k\geq 0$ such that $k||x||\leq||T(x)||$ $\forall x \in N$

Proof : Assuming that $k\ ||x|| \leq ||T(X)||$ $\forall x \in N$ and $k\geq 0$ ------------------(1)is true ,we claim that T^{-1} exists and is continuous.

By definition of inverse mapping T^{-1} exists $\Leftrightarrow$Tis one –one .Taking $x_1,x_2 \in N$,we have $T(x_1) = T(x_2) \Rightarrow T(x_1) - T(x_2)=0 \Rightarrow T(x_1 - x_2)=0 \Rightarrow (x_1 - x_2) =0 \Rightarrow$ $x_1 = x_2 \Rightarrow$Tis one-one and T^{-1} exists $\Rightarrow$there exists $x \in N$ corresponding to each y in the domain of T^{-1}such that $T(x) = y \Leftrightarrow T^{-1}(y) = x$-----------------------(2)

in view of(2), (1) can be written as $k\ ||T^{-1}(y)|| \leq ||y|| \Rightarrow ||T^{-1}(y)|| \leq$ $1/k\ ||y|| \Rightarrow T^{-1}$ is bounded and hence continuous.

Conversely if T^{-1} exists and continuous on its domain T{N] then $\forall x \in N$ there exists $y \in T(N)$ such that $T^{-1}(y) = x \Leftrightarrow T(x) = y$ ice Tis one-one. Now T^{-1} being continuous , it is bounded and so there exists a positive constant k such that $||T^{-1}(y)|| \leq k||y|| \Rightarrow ||x|| \leq k\ ||T(x)|| \Rightarrow k||x|| \leq ||T(x)||$,for $k=1/k>0$

Theorem10: If M be a closed linear subspace of a normed linear space N and T be a natural mapping (homomorphism)of N onto N/M such that $T(x)=x+M$,then show that T is continuous (bounded)linear transformation with $||T|| \leq 1$

Proof: Given that M is closed and N/M is a normed linear space with the norm of a coset x +M in N/M such that ||x +M|| = inf { ||x + m|| :m∈ M},we claim that Tis linear. For any x,y ∈ N and α,ß being scalars ,we have T**(αx + ßy) =** {(αx + ßy)+M}= α(x+ M) + ß(y+M) = αT(x) + ßT(y) ⇒T is linear. Again we claim that Tis continuous, sine ||T (x)|| = ||x+M || =inf {||x+ m|| :m ∈M}≤ ||m ||if m=0,in particular ≤1.||x|| ∀x ∈N ,as 0∈MandMis a subspace of ⇒T is bounded with bounds 1⇒Tis continuous. Also ||T|| = Sup {||T(x)||: ||x|| ≤1} for x∈N≤1.

Theorem11 **:**If N,.N' are two normed linear spaces and Tis a continuous linear transformation of N into N' and if M is the null space (kernel) of T ,then show that T induces a natural linear transformation T' of N/M into N' and that ||T'| =||T||

Proof : Note that Ker T or Null space of T is defined by Ker(T)or

N (T) ={x: x ∈N, T(x)=0 },here N (T)=M. We first claim that M is closed ,since if x be a limit point of M, then there exists a sequence $\{x_n\}$ in M such that $x_n \rightarrow x$ .But T is continuous therefore $T(x_n) \rightarrow T(x)$,now $T(x_n) = 0$ ∀ n ⇒

T (x)=0 ⇒x ∈M⇒ M is closed

Thus M being a closed subspace of N, N/M is a normed linear space with the norm of a coset x +M is N/M such that||x+ M|| =inf {||x +m || : m∈ M} .Now defining T' $:N/M \rightarrow N'$ and setting T' $(x +M)=T(x)$,we claim that $T'$is a linear transformation such that **$||T'| =||T||$.** Taking two elements x+ M and

y+ M of N/M and α, ß any scalars we have $T'[\alpha(x+ M) + \beta(y+ M)] =T'[(\alpha x+ \beta y)+ M]$ $= \alpha T'(x+ M) + \beta T'(y+ M) \Rightarrow T'$is linear. and $||T|| = \text{Sup}\{||T(x)||: ||x|| \leq 1\}, x \in N$ $= \text{Sup}\{||T(x)||:\inf ||x+ M||: m\in M \leq 1\}, x\in N = \text{Sup}\{||T(x)||:\inf ||x+ m||: \leq 1\}$ $x\in N, m \in M$ since $m \in M \Rightarrow T(m) =0 = \text{Sup}\{||T(x+ m)||:||x|| \leq 1\}=T$ as $x\in N$, $m \in M \Rightarrow x+m \in N$ and $x\in N \Rightarrow x+0 \in N$ for $0\in M$

Theorem 12: If N & N' are normed linear spaces ,then the set B (N,N')of all continuous linear transformations of N into N' is itself a normed linear space with respect to the point wise linear operations and the norm defined by $||T||= \text{Sup}\{||T(x)||: ||x|| \leq 1\}$ Further , if N' is a Bnach space , then B (N,N') is also a Banach space.

Proof : Since a set ζ of all linear transformations from a normed space N into normed space N' is itself a linear space w.r.t. point wise linear operations ,therefore to show that B (N,N') is a linear space .We claim that B (N,N') is a subspace of ζ .

$T_1 ,T_2 \in B (N,N') \Rightarrow T_1 ,T_2$ are bounded) ⇒there exists real k_1 , $k_2 \geq 0$ such that for all $x\in N$, $|| T_1(x)|| \leq K_1 ||x||$,and $|| T_2(x)|| \leq K_2 ||x||$, for scalars α, β, we have $||(\alpha T_1 + \beta T_2)(x)|| \leq ||(\alpha T_1(x)|| + ||\beta T_2(x)|| \leq$

$|\alpha| \; ||T_1(x)|| + |\beta| \; ||T_2(x)|| \leq |\alpha| K_1 ||x|| + |\beta| K_2 ||x|| \leq (|\alpha| K_1 + |\beta| K_2) ||x||$

showing that $\alpha T_1 + \beta T_2$ is bounded $\Rightarrow \alpha T_1 + \beta T_2 \in B(N,N') \Rightarrow B(N,N')$is a linear subspace of ζ.

Now we claim that B (N,N') is a normed linear space ,since

1) $||T|| = \text{Sup}\{||T(x)||: ||x|| \leq 1\}$ and $||T(x)|| \geq 0 \Rightarrow ||T|| \geq 0$

2) $||T|| = 0 \Leftrightarrow \text{Sup}\{||T(x)||: ||x|| \leq 1\} = 0, \quad \forall x \in N \Leftrightarrow \text{Sup}\{||T(x)|| \div ||x|| : x \neq 0\} = 0, \forall x \in N \Leftrightarrow ||T(x)|| = 0, \forall x \in N \Leftrightarrow T = 0$

3) $||T+S|| = \text{Sup}\{||T(x) + S(x)||: ||x|| \leq 1\} \leq \text{Sup}\{||T(x): ||x|| \leq 1\} + \{||S(x)||: ||x|| \leq 1\} \leq ||T|| + ||S||$.

4) $||(\alpha T|| = \text{Sup}\{||(\alpha T)(x)||: ||x|| \leq 1\} = |\alpha| \text{Sup}\{||T(x)||: ||x|| \leq 1\} = |\alpha| \, ||T||$

Again we claim that B (N,N')is complete, if N 'is complete. Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in B (N, N').$\{T_n\}_{n=1}^{\infty}$ we have $||T_n - T_m|| \to 0$ as $m,n \to \infty$ --- (1)

For each $x \in N$ $||T_n(x) - T_m(x)|| = |(T_n - T_m)(x)|| \leq ||(T_n - T_m)|| \; ||(x)|| \to 0$---(2)

hence for each $x \in N$, $\{T_n(x)\}$ is a Cauchy sequence in N', but since N' is complete there exists a vector $T(x)$ in N', such that $\{T_n(x)\} \to T(x)$ which defines a mapping T of N into N'. To show that T is a linear operator on N into N', for arbitrary scalars α, β and $x, y \in N$ we have $T(\alpha x + \beta y) = \text{Lim}_{n\to\infty} T_n(\alpha x +$

ßy) = α $\lim_{n\to\infty} T_n(x)$ + ß $\lim_{n\to\infty} T_n(y)$ = α T(x) + ß T(y) which shows that Tis linear

Again to show that T is bounded, we have $\|T(x)\| = \|\lim T_n(x)\| \leq$ $\lim \|T_n\| \, \|x\| \leq \sup(\|T_n\| \, \|x\|) = \sup(\|T_n\| \, \|x\|)$ -------------(4)

In view of (1) we observe that $|\,\|T_n - T_m\|\,|\,| \leq \|T_n - T_m\| \to 0$ as $m,n\to\infty \Rightarrow \{T_n\}$ is a Cauchy sequence of real numbers and so it is convergent and bounded $\Rightarrow$there exists $K \geq 0$ such that $\sup\|T_n\| \leq K$ -------------------------------(5) therefore (4) & (5) $\Rightarrow \|T(x)\| \leq K \Rightarrow$ T is bounded i.e. T∈ B (N, N'). Further we claim that $T_m \to T$, given $\varepsilon > 0$ and $\{T_n(x)\}$ is a Cauchy sequence ⇒ there exists a positive integer n_0 such that $n, m \geq n_0 \Rightarrow \|(T_n - T_m)\| < \varepsilon$ ---------------------- (6) ⇒

$\|T_n(x) - T_m(x)\| \leq \|(T_n - T_m)\| \, \|x\| < \varepsilon \|x\|$, forall $n, m \geq n_0$ and any vector x∈N

Proceeding the limit as $n\to\infty$,we find that $\lim_{n\to\infty} \|T_n(x) - T_m(x)\| =$

$\|T(x) - T_m(x)\| = \|(T - T_m)(x)\| \leq \varepsilon \|x\|$ --------------------------------- (7)

Since $\lim_{n\to\infty} T_n(x) = T(x)$ as the norm is a continuous function and

$\lim_{n\to\infty}\|T_n(x)\| = \|T(x)\|$, (7) $\Rightarrow \|(T - T_m)\| = \sup\{\|(T - T_m)\| \div \|x\| : x \neq 0\}$ $< \varepsilon$ for all $m \geq n_0$,by def. of $\|T\| \Rightarrow \|(T - T_m)\| \to 0$,as $m\to\infty \Rightarrow T_m \to T$ as $m\to\infty$,hence B (N, N'). is complete as N' is complete.

LECTURE -3

Next we will discuss a lemma & Hahn –Banach Theorem

Lemma2 : Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M .If x_0 is a vector not in M ,and if $M_0 = M + x_0$ is the linear subspace spanned by M and x_0 ,then f can be extended to a functional f_0 defined on M_0 such that $|| f || = || f_0 ||$.

Proof : We firs prove lemma under the assumption that N is a real normed space. Since x_0 is not in M ,each vector m in M_0 is uniquely expressible in the form $m = x + \alpha x_0$ with x in M .It is clear that the definition $f_0(m) = f_0(x + \alpha x_0) = f_0(x) + \alpha f_0(x_0) = f(x) + \alpha r_0$ extends f linearly to M_0 for every choice of real number $r_0 = f_0(x_0)$ also f_0 is an extension of f ,for if $x \in M$, then $x = x + 0x_0$, so that $f_0(x) = f_0(x + 0x_0) = f(x) + 0\, r_0 = f(x)$ ------ (1) so f_0 is an extension of f over M.

Now we claim that $|| f || = || f_0 ||$.

$|| f_0 || = \text{Sup}\{|f_0(x)| : ||x|| \leq 1], x \in M_0 \geq \text{Sup}\{|f_0(x)| : ||x|| \leq 1\}, x \in M$ as

$M \subset M_0 = \text{Sup}\{|f_0(x)| : ||x|| \leq 1\}, x \in M$ and $f_0 = f$,hence $|| f_0 || \geq ||f||$ ------(2)

We now observe that for any two vectors x_1, x_2 in M we have, $f(x_2) - f(x_1)$

$= f(x_2 - x_1) \leq |f(x_2 - x_1)| \leq ||f||\, ||(x_2 - x_1)|| = ||f||\, ||(x_2 - x_0 + x_0 - x_1)|| =$

$||f||\, ||(x_2 + x_0|| + ||f||\, ||(x_0 + x_1)||$ or $- f(x_1) - ||f||\, ||(x_1 + x_0)|| < -f(x_2) +$

$||f||\,||(x_2+x_0\,||$ ------------(3) .If we define two real numbers a & b $\forall$ $y\epsilon$ M by

a=Sup $\{-f(y-||f||\,||(y+x_0)||: y\,\epsilon M\}$and b=inf $\{-f(y)+||f||\,||(y+x_0||: y\,\epsilon M\}$

then (2) shows that $a \le b$.If we now choose r_0 to be any real number such

that $a\le r_0 \le b$,then for arbitrary $m = x+\alpha x_0$ and setting $y = x/\alpha$,we find

$-f(x/\alpha) - ||f||\,||(x/\alpha + x_0)|| \le r_0 \le - f(x/\alpha) + ||f||\,||(x/\alpha + x_0)||$ ------------- (4)

for $\alpha > 0$ the last two part of inequality (3) yield $r_0 \le -1/\alpha\, f(x)+$

$1/\alpha\, ||f||(x+\alpha x_0) \Rightarrow f(x) + \alpha r_0 \le ||f||\,||(x+\alpha x_0)|| \Rightarrow f_0(x) + \alpha f_0(x_0) \Rightarrow$

$f_0(x+\alpha x_0) \le ||f||\,||(x+\alpha x_0)|| \Rightarrow f_0(m) \le ||f||\,||m||$ ----------------------- (5)

similarly we can prove for $\alpha < 0$ that $f_0(m) \le ||f||\,||m||$ ------------ (6) thus

from (5) & (7) we have $f_0(m) \le ||f||\,||m||$ -------------------------------(7) &

from (1) for $\alpha = 0$ $||f_0|| = ||f||$. If we replace m by –m in (7) we get

$f_0(-m) \le ||f||\,||-m|| \Rightarrow -f_0(m) \le ||f||\,||m||$ -------------------------(8)

(7) & (8) $\Rightarrow |f_0(m)| \le ||f||\,||m||) \Rightarrow ||f_0|| \le ||f||$ --------------------------(9)

Thus from (1), (2) & (9) we have $||f_0|| = ||f||$, hence proved.

We next use the result of the above lemma to prove for the case in which N is complex. Here f is a complex valued functional defined on M for which $||f||=1$ .Complex linear space can be regarded as a linear space by simply restricting the scalars to be real numbers. If g & h are real & imaginary parts of f, so that $f(x) = g(x) + i\,h(x)$ $\forall x\,\epsilon M$, then both g & h are easily seen to be real functional on the real space M, and since $||f||=1$,we have $||g|| \le 1$ .The equation

$f(ix) = if(x)$together with $f(ix)=i\,g(x)+i\,h(x) = i\,g(x)- i\,h(x)$ shows that $h(x) =-g(ix)$,so we can write $f(x) =g(x) - h(x)$.Now we can extend g to a real valued functional g_0 on the real space M_0 in such a way that $||g_0|| = ||g||$ and we define f_0 for x in M_0 by $f_0(x) = g_0(x) - i\,g_0(ix)$.It is easy that f_0 is an extension of f from M toM_0 ,that $f_0(x+y) =f_0(x)+ f_0(y)$ and$f_0(\alpha x)=\alpha f_0(\alpha x)$ for all real α's. The fact that the property is also valid for all complex α's is a direct consequence of $f_0(ix)= g_0(ix)- i\,g_0(i^2x)= i\{ g_0(ix)- i\,g_0(ix)\}=i\,f_0(x)$,so f_0 is linear as a complex valued function defined on the complex space M_0. .All that remain to be proved is that $||f_0|| =1$,and we dispose of this by showing that if x is a vector in M_0 for which $||x||=1$then $|f_0(x)| \leq 1$.If $f_0(x)|$is real ,it follows from $f_0(x) = g_0(x)$ and $||g_0|| \leq 1$.If $f_0(x)$ is complex ,then we can write $f_0(x) = re^{i\theta}$ with $r>0$,so$|f_0(x)| =r = e^{-i\theta} f_0(x)= f_0(e^{-i\theta} x)$ our assumption follows from $|| e^{-i\theta} x|| = ||x|| =1$ and the fact that $f_0(e^{-i\theta} x)$ is real.

Theorem 13: (The Hahn –Banach Theorem) : let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. Then f can be extended to a functional f_0 defined on the whole space N such that $|| f_0 || = ||f||$

Proof : In view of above lemma for any $x\in N$ but x does not belong to M ,we can have an extension of f on M U {x} such that $||f||$ is conserved for extension. If we consider the set of all possible extension of f on all the subspaces

M U {elements of N not in M}of N containing M, then the set of extension of f say G can be partially ordered as under, taking $g_1, g_2 \epsilon$ G and relation ≤ such that $g_1 \le g_2 \Rightarrow$domain of g_1 is contained in domain of g_2 and $g_1 (x) = g_2 (x) \forall \epsilon$ Dom .(g_1). Then obviously (G, ≤) is partially ordered ,since it is reflexive, antisymmetric and transitive. Also we observe that the union of any chain of extensions is an extension and is therefore an upper bound for the chain. Thus every chain in G has an upper bound. As such from Jorn's lemma there exists a maximal extension $f_0 \epsilon G$,otherwise there exists an xϵ N and x does not belong to M such that f_0 can be extended to domain of f_0U{x} i.e. M U {x} by above lemma .But this violets the maximality of f_0 .Hence the domain of f_0 must be the whole space N such that $|| f_0 || = ||f||$.

LECTURE-4

Today we will discuss some theorems based on HAHN –Banach Theorem

Theorem14 : If N is a normed linear space and x_0 is a non zero vector in N then there exists a functional f_0 in N * such that $f_0(x_0) = ||x_0||$ and $|| f_0 ||= 1$

Proof : Let $M= \{\alpha x_0\}$ be the linear subspace of N spanned by x_0 AND DEFINE F on M by $f(\alpha x_0)= \alpha ||x_0||$.It is clear that f is a functional on M such that $f(x_0) = ||x_0||$ and $|| f ||= 1$. By the Hahn -Banach Theorem f can be extended to a functional f_0 in N* with the required properties. This result shows that N* separates the vectors in N, for if x and y are any two distinct vectors ,so that (x –y) not equal to zero ,then there exists a functional f in N* such that f(x –y)not equal to zero or equivalently f(x) not equal to f(y)

Theorem15If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M ,then there exists a functional f_0 in N* such that $f_0(M)=0$ and $f_0(x_0)$ not equal to zero.

Proof : The natural mapping T of N onto N/M is a continuous linear transformation such that T(M) =0 and $T(x_0)= x_0 +M$ not equal to zero. By above theorem, there exists a functional f in (N/M)* such that $f(x_0 +M)$ not equal to zero. If we define f_0 by $f_0(x) = f(T(x)$,then f_0 is easily seen to have the desired properties.

Theorem16 :If Mbe a closed linear subspace of a normed linear space N and x_0 be a vector in N ,but not inM with the property that the distance from x_0 to M ie $d(x_0 ,M)=d>0$then there exists a bounded linear functional F$\epsilon\tilde{N}$such that

$||F|| =1$

$F (x_0) =d$ and $F(x) =0$ $\forall$ xϵ M ie. $F(M) =0$,we have by definition

$d =\inf \{||x_0 -x||: x\epsilon M\}$, $d>0$.--------(1) Consider $M_0 = MU \{0\}$as a subspace spanned by M and x_0. Any yϵ M_0 can be written as $y = x +\alpha x_0$ -------------(2),so that $M_0 =\{ x +\alpha x_0 : x\epsilon M, \alpha \text{ is real}\}$ -- (3) where M is closed and x_0 does not belong to M.

Define a mapping f_0 on M_0 by $f_0(y) = \alpha d$,---------------- (4) clearly y is unique and f_0 is linear on M_0. Now $f_0(x_0)= f_0(0+1 x_0)=1.d = d$, by (4)and for any m ϵM $f_0 (m)= f_0(m+0 x) =0d =0 \Rightarrow f_0 (M) =\{0 \}$ Now we claim that$|| f_0 || =1$,since $||f_0||=\text{Sup} \{|f_0 (y)|\div||y|| : y \text{ not equal to } 0\}$, $y\epsilon M_0= \text{Sup} \{|f_0 (x+\alpha x_0)|\div|| x+\alpha x_0 || : x \,\&\, \alpha \text{ not equal to } 0\}$, $x \epsilon M,\alpha\epsilon R$ $=\text{Sup}\{ |\alpha d|\div|| x+\alpha x_0||: \alpha \text{ not equal to } 0\}$

$= \text{Sup}\{ d\div|| x_0+x/ \alpha ||: \alpha \text{ not equal to } 0\}$, $x \epsilon M,\alpha\epsilon R$ as $d>0=$

$d \text{ Sup}\{1/ ||x_0 -w||\} =d[\inf \{||x_0 -w||\}]^{-1} =d.1/d =1$ so f_0 is a linear functional on M_0 such that $f_0 (M) =\{0 \}$, $f_0(x_0)=d$ and $|| f_0 || =1$ ------------------- (5)

in view of Hahn-Banach Theorem there exists a functional F on the whole space N such that $F(y)=f_0 (y)$ $\forall y\epsilon M_0$ and $||F|| = || f_0 ||$so that by (5) $F(M) =0$, $F(x_0) =d$ and$||F|| =1$.

LECTURE -5

Now we shall discuss the Dual spaces Natural imbedding Separability **and Weak topology** *in Normed spaces*

If N be a normed space ,then the set of all bounded linear functionals defined on N form a banach space, denoted by **N* and known as Dual or Conjugate space or adjoint space or the first dual space of the normed space N.** The space of bounded linear functionals on N*is known as the **second dual space of N** and **denoted by N****

Taking N* and N** as the first and second conjugate spaces of a normed linear space N so that each vector x in N gives rise to a functional f or function f(x) in N* and a functional F_x in N**,we define F_x as $F_x(f) = f(x)\ \forall\ f \in N^*$ ----------(1)

clearly F_x is linear ,since by (1) $F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x)$,f, g ϵ N*and α,ß are scalars $= = \alpha f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g)$ --------------------------------(2)

and $||F_x|| = \text{Sup}\{|F_x(f)| : ||f|| \leq 1\}, f \in N^* \leq \text{Sup}\{ ||f||\,||x|| : ||f|| \leq 1\} = ||x||$----(3)

The mapping $J : N \rightarrow N^{**}\ \forall\ x \in N$ ie $\mathbf{J(x) = F_x}$ **or** $\mathbf{J : x \rightarrow F_x}$ with $\mathbf{F_x(f) = f(x)}$ $f \in N^*$ is norm preserving mapping of N into N** and known as **the Natural imbedding or canonical mapping of N into N****

Now the mapping $J : x \rightarrow F_x$ is linear and so is an **isometric isomorphism of N into N**** since $\forall f \in N^*$ we have $F_{x+y}(f) = f(x+y) = f(x) + f(y) = F_x(f) + F_y(f) = (F_x + F_y)(f)$-------------------- (5) and $F_{\alpha x}(f) = f(\alpha x) = (\alpha F_x)(f)$-------------- (6)

Here J is one-one since for any x, y ϵN we have J (x) =J(y) ⇒ $F_x(f)= F_y(f)$

∀ f ϵ N* ⇒ f(x)= f(y) ⇒ f(x)- f(y) =0 ⇒f (x-y) =0 ⇒x-y =0 ⇒x=y -----------(7)

if the natural imbedding or J : x→ F_x of N into N**is onto then we called the normed space N as **reflexive**. If the range i.e. **R(J) =N****then the normed space Nis called **algebraically reflexive.**

The normed space **N is separable** if it contains a **denumerable or countable** dense subset i.e. A normed space **is separable** if there exists a sequence $\{x_n\}$ϵN such that ,there correspond an element x_{n0} of $\{x_n\}$for every x ϵN and for every ε >0 with $||x - x_{n0}||< \varepsilon$

The set R,R^n, C[0,1] andC^∞ are seperable.

The **weakest topology of N** w.r.t. which all **elements of N*remain continuous** is known as **Weak topology on N.**

Theorem 17: Every subset of separable normed space is separable

Proof : Assuming that A is a subset of a separable normed space N and B= $\{x_n\}$ is a countable dense set in N, we claim that A is separable since for a numerical sequence $\varepsilon_n \to 0$ $\varepsilon_n>0$,we can find for each i=1,2------n a_{in} ϵA such that $||x_i - a_{in}||< \inf a_{in} \epsilon A \, || x_i - y_{in} ||+ \varepsilon_n$ -----------------(1)if we take x ϵA and ε>0 ,then there exists an x_i ϵB such that $||x- x_i||<\varepsilon/3$. Now taking n sufficiently large such that $\varepsilon_n <\varepsilon/3$,we have$||x - a_{in}||= ||x - x_i +x_i - a_{in}||\leq$

$||x - x_i|| + ||x_i - a_{in}|| \leq \varepsilon/3 + \inf.||x_i - a_{in}|| + \varepsilon_n \leq \varepsilon/3 + ||x_i - x|| + \varepsilon_n$, as x, $a_{in} \in A$ and x is arbitrary $< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. This shows that the sequence $\{ a_{in} \} \in A$ and hence A is separable.

Theorem18: If N is a normed Linear space, then the closed unit sphere S^* in N^* is compact Hausdorff space in the weak topology on N*

Proof. We already know that S*. is a Hausdorff space in this topology, so we confine our attention to proving compactness. With each vector x in N we associate a compact space $C_{x1}$where C_x is the closed interval $[-\|x\|, \|x\|]$or the closed disc $[z: |z| \leq \|x\|]$, according as N is real or complex. By Tychonoff's theorem, the product, C of all the C_x's is

When we use the adjective "closed" in referring to S^*_1 we intend only to emphasize the inequality $\|f\|] \leq 1$, as contracted with $\|f\|] < 1$, also a compact space. For each x, the values f (x) of all f's in S^*lie in C_x. This enables us to imbed S^* in C by regarding each f in S^*as identical with the array of all its values at the vectors x in N. It is clear from the definitions of the topologies concerned that the weak* topology on S^* equals its topology as a subspace of C; and since C is compact, it suffices to show that S^* is closed as a subspace of C. We show that if q is in $\overline{S^*}$, then since q is in C we have $|g(x)| \leq \|x\|$ for every x in N. It therefore suffices to show that q is linear as a function defined on N. Let $\in > 0$ be given, and let x and y be any two vectors in N. Every basic neighborhood of g intersects S^*, so there exists an f in S^* such that $|g(x)-f(x)|<\in/3$, $|g(y)-f(y)|<\in/3$, and $|g(x+y)-f(x+y)|<\in/3$. Since f is linear, f (x-y) -f (x)-f (y) =0, and we therefore have

$$|g(x+y)-g(y)|=|g(x+y)-f(x+y)|-[g(x)-f(x)]$$

$$-[g(y)-f(y)\ ]\leq|g(x+y)-f(x+y)|+|g(x)-f(x)|$$

$$+|g(y)-f(y)\,|<\in/3+\in/3+\in/3=\in.$$

The fact that this inequality is true for every $\in>0$ now implies that g (x+y) = g (x) + g (y). We can show in the same way that

$$g(\alpha x) = \alpha g(x)$$

for every scalar α, so g is linear and the theorem is proved.

Theorem 19: Every finite dimensional normed linear space is algebraically reflexive.

Proof : We have already defined that an isomorphic mapping J from N into N** is a linear transformation from N into N** and also J is one-one. Moreover Nis finite dimensional so that dim. N= dim.N * = dim. N**, therefore J is onto le. range of J, R(J) = N** ⇒N is algebraically reflexive.

LECTURE -6

now we will discuss about ***L_p or L^p pace and Holder.s Minkowski inequality in L_p Space.***

L_p or L^p Space : L_p -space consists of all measurable function f defined on a measure space X with measure μ such that ∫ $|f(x)|^p$ x ∈X is integrable with the norm $||f||_p = \{\int|f(x)|^p \, d\,\mu(x)\}^{1/p}$ or simply $\{\int |f|^p \, d\,\mu\}^{1/p}$ .Here$||f||_p$ is known as the L_p- norm of for a norm of f. and $1 \leq p < \infty$.

Theorem20 :Holder's inequality for L_p space: If f & g ∈ L_p with $p>1$ and $1/p+1/q =1$,then f ,g ∈ L_1 $||fg||_1 \leq ||f||_p \, ||g||_q$ or $\int |fg| \leq \{\int |f|^p\}^{1/p}\} \{\int |g|^q\}^{1/q}$

Proof : We have by ***cor1 of lemma1*** that if a, b ≥ 0 ,p>1 and 1/p+1/q=1 then

$ab \leq a^p/p + b^q/q$ with sign of equality iff $a^p = b^q$ ---------------(1) .If f,& g =0 then it is trivial

Taking $||f||_p$ & $||g||_q$ not equal to zero and setting $a = |f(x)|/||f||_p$ and

$b = |g(x)|/ ||g||_q$ product f g being measurable we have by inequality(1)

$|f(x)|/||f||_p . |g(x)|/||g||_q \leq 1/p \, |f(x)|^p/||f||^p_p + 1/q \, |g(x)|^q/ ||g||_q^q$ ------------(2)

But f∈ L_p ,$||f||^p_p \Rightarrow \int|f|^p < \infty \Rightarrow |f|^p$ is integrable and similarly $|g|^q$ is integrable so that by (2) |f| |g| =|f g| is integrable .Thus f &g ∈L_1. Integration of (2)yields

$1/||f||_p \, ||g||_q \int |f| \, |g|$ ie$1/ ||f||_p \, ||g||_q \int |fg| \leq 1/p/||f||^p_p \int|f|^p + 1/q||g||_q^q \int|g|^q$

0r, $||f|| \, ||g||/ ||f||_p \, ||g||_q \leq 1/p + 1/q$ as $\int|fg| = ||fg||_1, \int|f|^p = ||f||^p_p, \int|g|^q = ||g||_q^q$

or $||fg||_1 \leq ||f||_p||g||_q$ ---------------------------(3) which may also be written as

$\int|fg| \leq \{\int |f|^p \}^{1/p}\} \{\int |g|^q \}^{1/q}$ --------------------(4)

Cor2 .: If p= q=2 then (3) reduces to CauchySchwarz inequality as

$||fg||_2 \leq ||f||_2||g||_2$ ---------------------------------(5)

Theorem 21: Minkowski Inequality for L_p space : If f & g $\in L_p$ with p>1 and $1/p+1/q=1$,then $||f+g||_p \leq ||f||_p + ||g||_p$

Proof: The case p=1 is evident by norm –axiom as follows $\int|f+g| \leq \int|f|+|g|$ i.e. $\int|f|+\int|g| \Rightarrow ||f+g||_1 \leq ||f||_1 .||g||_1$ ---------------------(1) Taking case p>1 ,we have

$|f+g|^p \leq \{|f|+|g|\}^p \leq [2 \max. \{|f|,|g|\}]^p \leq 2^p \max \{|f|^p + |g|^p\}$, f & g $\in L_p$ ----------(2)

Integrating both sides

$\int|f+g|^p \leq 2^p \int|f|^p + 2^p \int |g|^p\} < \infty \Rightarrow f+g \in L_p$ --(3)

Again $|f+g|^p = |f+g| . |f+g|^{p-1} \leq |f| |f+g|^{p-1} + |g||f+g|^{p-1}$,so that

$\int|f+g|^p = |f+g| . |f+g|^{p-1} \leq \int|f| |f+g|^{p-1} + \int |g||f+g|^{p-1}$ -----------------------------(4)

Also $|f+g| \in L_p \Rightarrow |f+g| \in L_1$, $/p+1/q=1 \Rightarrow p = (p-1) .q$ and $|f+g|^{p-1} = |f+g|^{p/q} \in L_q$

On applying holder's inequality on the right side of (4)we get

$\int |f+g|^p \leq ||f||_p ||(|f+g)|^{p-1}||_q + ||g||_p ||(|f+g)|^{p-1}||_q$

0r $||f+g||_p^p \leq ||f||_p \{\int(|f+g)|^{(p-1)q}\}^{1/q} + ||g||_p \{\int(|f+g)|^{(p-1)q}\}^{1/q} \leq$

$\{||f|_p + ||g||_q\} ||f+g||^{p/q}_p$,or $||f+g||_p \leq ||f||_p + ||g||_p$,as $p - p/q = 1$ ----------(5)

LECTURE -7

In this lecture we shall study about open mapping, projection and closed graph in normed linear space also theorems based on these.

If X & Y are two topological spaces then a map f : X→Y is known as ***open mapping if ∀open set V of X, the set f(V) is open in Y.***

In Banach spaces B, B'the open spheres with radius r and centre at x are denoted respectively S (x, r)or S_r (x) and S' (x, r)or S_r '(x) whereas the open spheres in B, B 'with radius r and centre at origin are denoted respectively by S, or S_r and S'or S_r '

Also we have S (x, r) or $S_r(x) = x + S_r$ ------------ (1) and $S_r = r S_1$------------- (2)

Since y ϵ S (x, r) ⇔ ||y −x||<r by def. of open sphere⇔||u||<r ⇔y= x +u and ||u||<r ⇔ y ϵ x + S_r and S_r = [x: ||x||<r] ={x: ||x||/r <1}=[rz: ||z||<1] = r S_1.

In a linear space L, a linear operator T: L→ L is **idempotent if T^2 =T** and ***nilpotent if T^2 =0*** .If M be a subspace of L then M is ***T-variant under T if T maps M into itself*** i.e. if m ϵM⇒ T(m) ϵM⇒ T(m) ⊂ M. The linear space ***L is the direct sum of its subspaces M ,***N denoted by L = M ⊕ N.,if any l ϵL is uniquely expressible as

l = m + n , , m ϵM ,n ϵN m ϵM, the result being generalized for any number of subspaces. The linear operator Tis said to be ***reduced*** by the pair ***(M,N)*** if

L= M $\oplus$ N and both M &N are T-varient.--------------------- (3)

The projection E on M along Nis a map E :L→ L such that E(l) =E (m+n) =m with l=m+n--(4)

Criteria for E to be a projection is that a linear transformation E on L is a projection on some subspace iff E is idempotent i.e. E^2 =E or

E is a projection ⟺ E^2 =E ------------------------ (5)

If L= $M \oplus N$. and E is a projection on M along N, then Range of E i.e. R (E)=M and Null Space of E i.e N_0 (E)=N ----(6) i.e.={E(x) : x∊L} and N={x:E(x)=0}--- (7)

A projection on a Banach space B is an ***idempotent linear operator P on B***,which ***is continuous*** i.e. P is a projection on B if ***P^2= P and P is continuous.***

Closed graph Transformation If X &Y are two non empty sets and f :X → Y is a mapping with domain X and range in Y then the graph of f denoted by f_G is the subset of X ×Y which consists of all ordered pairs of the form (x, f(x)) i.e. if D be a subset of X and f :D→Y,then the graph of Tis defined as f_G={ x,f(x)}:x ∊D}---(8)

In case of two normed linear spaces N, N' with D⊂ N and T:D→Y,then the graph of T is given by T_G = {X, T(x):x∊D} -----------------(9)

Theorem22 : If B and B' are Banach spaces, and if T is a continuous linear transformation of B on to B', then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'.

Proof: We denote by S_r and $S_r^{'}$ the open spheres with radius r centered on the origin in B and B'. It is easy to see that

$$T(S_r) = T(rS_r) = rT(S_1),$$

on it suffices to show that T (S_1) contains some S_r^1.

We begin by proving that $\overline{T(S_1)}$ contains some S_r^1. Since T is on to, we see that $B^{'} = \cup_{n=1}^{\infty} T(S_n)$.B' is complete, so Baire's theorem implies that some $\overline{T(S_{n_0})}$ has an interior point y_0, which may be assumed to lie in $T(S_{n_0})$, The mapping $y \rightarrow y - y_0$, is a homeomorphism of B' on to itself, so $\overline{T(S_{n_0})}$-$y_0$has the origin as an interior point. Since y_0 is in $\overline{T(S_{n_0})}$-y_0= $\overline{T(S_{n_0}) - y_0} \subseteq \overline{T(S_{2n_0})}$, which shows that the origin is an interior point of $\overline{T(S_{2n_0})}$. Multiplication by any non-zero scalar is a homeomorphism of B' onto itself, so $\overline{T(S_{2n_0})}$= $\overline{2n_0 T(S_i)} = 2n_0 \overline{T(S_i)}$; and it follows from this that the origin is also an interior point of $\overline{T(S_i)}$; so $S_{\in}^{'} \subseteq \overline{T(S_1)}$ for some positive number $\in$.

We conclude the proof by showing that $S_{\in/3}^{'} \subseteq \overline{T(S_1)}$. Let y be a vector in B' such that $\|y\| < \in$.Since y is in $\overline{T(S_1)}$;, there exists a vector x_1 in B such that $\|x_1\|$ <1 and $\|(y - y_1) - y_2\| < \in / 2$, where $y_1 = T(x_1)$. We next observe that $S_{\in/2}^{'} \subseteq \overline{T(S_{1/2})}$; so there exists a vector x_2 in B such that $\|x_2\|$ <1/2 and $\|(y-y_1)-y_2\|<\in/4$, where y_2 = T (x_2). Continuing in this way, we obtain a sequence {x_n} in B such that $\|x_n\|<1/2^{n-1}$ and $\|y-(y_1+y_2+....+y_n)\|<\in/2^n$, where y_n = T (x_n), If we put

$$S_n = x_1 + x_2 + ... + x_n,$$

then it follows from $\|x_n\|<1/2^{n-1}$ that {s_n} is a Cauchy sequence in B for which

$$\|S_n\| \leq \|x_1\|+\|x_2\|+......+\|x_n\| < 1+1/2+...+1/2^{n-1} < 2.$$

Be is complete, so there exists a vector x in B such that $S_n \to x$; and $\|x\| = \|\lim S_n\| = \lim\|S_n\| \leq 2<3$ shows that x is in S_3. All that remains is to notice that the continuity of T yields.

$T(x) = T(\lim S_n) = \lim T(S_n) = \lim(y_1+y_2+....+y_n) = y$, from which we see that y is in T (S_3).

Theorem 23: (Open mapping Theorem) If B,B' are Banach spaces and if Tis a *continuous linear transformation of B onto B' ,then Tis an open mapping.*

Proof: We must show that if G is an open set in B, then T (G) is also an open set in B'. If y is a point in T(G), it suffices to produce an open sphere centered on y and contained in T(G).Let x be a point in G such that T(x) =y. Since G is open, x is the center of an open sphere which can be written in the form x + S,- contained in G. Our lemma now implies that T(S_r) contains some S'_{r_1}. It is clear that y $+S'_{r_1}$ is an open sphere centered on y, and the fact that it is contained in T(G) follows at once from $y + S'_{r_1} \subseteq y + T(S_r) = T(x) + T(S_r) = T(x + S_r) \subseteq T(G)$.

Theorem24: If *P* is a projection on a Banach space *B*, and if *M* and *N* are its range and null space, then *M* and *N* are closed linear subspaces of *B* such that *B*= M $\oplus$ N.

Proof: *P* is an algebraic projection, so (1) gives everything except the fact that *M* and *N* are closed. The null space of any continuous linear transformation is closed, so *N* is obviously closed and the fact that *M* is also closed is a consequence of

$$M = \{P(x): x \varepsilon B\} = \{x : P(x) = x\} = \{x : (1 - P)(x) = 0\}$$

which exhibits M as the null space of the operator I - P.

Theorem25: Let B be a Banach space, and let M and N be closed linear subspaces of B such that B = M $\oplus$ N. If z = x + y is the unique representation of a vector in B as a sum of vectors in M and N, then the mapping P defined by P (z) = x is a projection on B whose range and null space are M and N.

Proof: Everything stated is clear from definition of projectionandTh.9 except the fact that P is continuous, and this we prove as follows. if B' denotes the linear space B equipped with the nom defined by

$$\|z\|' = \|x\| + \|y\|,$$

then B' is a Banach space; and since $\| P(z) \| = \| x \| \leq \| x \| + \| y \| = \| z \|'$, P is clearly continuous as a mapping of B' in to B. It therefore suffices to prove that B' and B have the same topology. If T denotes the identity mapping of B' onto B, then

$$\| T(z) \| = \| z \| = \| x + y \| \leq \| x \| + \| y \| = \| z \|$$

shows that T is continuous as a one-to-one linear transformation of B' onto B. Theorem B now implies that T is a homeomorphism, and the proof is complete.

Theorem 26 :If Lbe the direct sum of two subspaces M &N ie $L = M \oplus N$ such that $M \cap N = \{0\}$ and an element $z \in L$is expressible uniquely as $z = x+y \quad x \in M, y \in N$ then show that projection Eon M along N defined by $E(z) = x$ is a linear operator

Proof :Given $E(z) = x$,$x \in M$,M being a subspace of $L \Rightarrow E: L \rightarrow L$ .Now if $z_1, z_2 \in L$ and a,b are scalars with $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$ where $x_1, x_2 \in M$ and $y_1, y_2 \in N$ then we have by def $E(z_1) = E(x_1 + y_1) = x_1$ and $E(z_2) = E(x_2 + y_2) = x_2$ -----(1)

Now $az_1 + bz_2 = a(x_1 + y_1) + b(x_2 + y_2) = \{a(x_1) + b(x_2)\} + \{(ay_1) + b(y_2)\}$

$\{a(x_1) + b(x_2)\} \in M, \{(ay_1) + b(y_2)\} \in N$ hence $E(az_1 + bz_2) = a\,E(z_1) + b\,E(z_2)$

$\Rightarrow$E is linear

Theorem 27:(the Closed Graph Theorem). If B and B' are Banach spaces, and if T is a linear transformation of B into B', then T is continuous $\Leftrightarrow$ its graph is closed.

Proof: In view of the above remarks, we may confine our attention to proving that T is continuous if its graph is closed. We denote by B_1 the linear space B renormed by $\| x \|_1 = \| x \| + \| T(x) \|$. Since

$$\| T(x) \| \leq \| x \| + \| T(x) \| = \| x \|_1,$$

T is continuous as a mapping of B_1 into B'. It therefore suffices to show that B and B_1 have the same topology. The identity mapping of B_1 onto B is clearly continuous, for $\| (x\| \le \| x \| + \| T(x)\| = \| x \|_1$. If we can show that B_1 is complete then Theorem B will guarantee that this mapping is a homeomorphism, and this will conclude the proof. Let (x_0) be a Cauchy sequence in B_1. It follows that $| x_0 |$ and $| T(x_0) |$ are also Cauchy sequences in B and B', and since both of these spaces are complete there exist vectors x and y in B and B' such that $\|x_0 - x\| \to 0$ and $\|(x_0) - y\| \to 0$. Our assumption that the graph of T is closed in B × B' implies that (x, y) lies on this graph so T (x) = y. The completeness of B_1 now follows from

$$\|x_0 - x\|_1 = \|x_0 - x\| + \| T(x_n - x)\| = \|x_0 - x\| + \|T(x_0) - T(x)\|$$

$$= \|x_0 - x\| + \| T(x_0) - y \| \to 0$$

The closed graph theorem has a number of interesting applications to problems in analysis, but since our concern here is mainly with matters of algebra and topology we do not pause to illustrate its uses in this direction.[1]

LECTURE -8

In this lecture we will study uniform boundedness Theoremand some theorems ,cor on conjugate space

We know that a collection C = {f_i: X→R}of real valued function defined on an arbitrary set X is known as ***uniformly bounded if there exists k ϵR*** such that

||f(x)|| ≤k ∀ f ϵ C and ∀ x ϵ X

Theorem 28:(the Uniform Boundedness Theorem). Let B be a Banach space and N a normed linear space. If (T_i), is a non-empty set of continuous linear transformations of B into N with the property that (T_i(x) is a bounded subset of N for each vector x in B, then $\{\|T_i\|\}$is a bounded set of numbers: that is, {T_i} is bounded as a subset of B (B N).

Proof: For each positive integer n the set

$$F_n = \{x : x \varepsilon B \text{ and } \|T_i(x)\| \le n \text{ for all } i\}$$

is clearly a closed subset of B, and by our assumption we have

$$B = \cup_{n=1}^{\infty} F_n,$$

Since B is complete, Baire's theorem shows that one of the F'$_n$s, say F_{n_0} has non-empty interior and thus contains a closed sphere S_0 with center x_0 and radius r_0 > 0. This says in effect, that each vector in every set $T_i(S_0)$ has norm less than or equal to n_0; and for the sake of brevity we express this fact by writing $\|T_i(S_0\| \le n_0$. It is clear that S_0- x_0 is the closed sphere with radius r_0 centered on the origin, so $(S_0 - x_0)/r_0$ is the closed unit sphere S. Since x_0 is in S_0, it is evident that $\|T_i(S_0 - x_0)\| 2n_0$. This yields $\|T_i(S)\| \le 2n_0/r_0$, so $\|T\| \le 2n_0/r_0$ for every i, and the proof is complete.

Theorem29 : If T be an operator on a normed linear space N and T* its conjugate defined by T*: N*→N*:T* (f)=f(T)and[T*(f)] (x)=f[T(x)] ∀f ϵN*and ∀x ϵN

then the mapping J : B(N) → B(N*);J (T) =T* ∀ T ϵB(N)is an isometric isomorphism of B(N) into B(N*)which reverses products and preserves the identity transformation.

Proof : Given T*: N*→N*:T* (f)=f(T) ∀f ϵN*--------------------------(1) where

[T*(f)] (x)=f [T(x)] ∀x ϵN--- (2)

We first claim that T* is linear ,since f, g ϵN*andα, β are scalars then

[T* (αf +βg)](x)= (αf +βg)][T (x)] by (2)= α[T*(f)] (x) +β[T*(g)] (x) by (1)

Again we claim that T* is unique let if possible there exists another conjugate T_0* of T ,so that T^*_0(f)= f (T) and T*(f) =f (T)   ∀f ϵN* -------------(3)

The definition of conjugate follows that [T^*_0(f)] (x)=f [T(x)]=[T*(f)] (x) ∀x ϵN

Hence [T^*_0(f)- T*(f)] (x)=0⇒ T^*_0=T*, Now we claim that T^* is bounded $||T^*||=\text{Sup}\{||T^*(f)||: |f||\leq 1, |x||\leq 1\}$

$.\leq \text{Sup}\{f[T(x)]:|f||\leq 1,|x||\leq 1\}\leq \text{Sup}\{||f||||T||||x||]: |f||\leq 1, |x||\leq 1\}\leq ||T||$,as$|f||\leq 1$, $|x||\leq 1$Thus $||T^*|| \leq ||T||$ ----------------------------(4) $\Rightarrow T^*$ is bounded as $||T||$is bounded $\Rightarrow T^*$ is an operator on N^*. ---------------(5)

But by **Th.(14)** for each non zero vector x in N there exists a functional f ϵN^*such that $||f||=1$ .in the present case T being an operator on N and T(x) a vector in N,there exists a functional f in N^* such that $f[T(x)]=||T(x)||$ with $||f||=1$------------------------------------- (6)

$||T||=\text{Sup}\{||T(x)||\div|| x||: x\neq 0\} = \text{Sup}\{|f[T(x)]| \div|| x||: ||f|| =1, x\neq 0\}=$

$\text{Sup}\{|T^*(f)(x)|: |f||=1, x\neq 0\}\geq \text{Sup}\{||T^*(f)||||x||: |f||=1, x\neq 0\}\geq ||T^*||$--(7)

(4) (4) & (7) $\Rightarrow ||T^*|| = ||T||$---(8)

Lastly to show that th mapping $J : B(N) \to B(N^*):J(T) = T^* \ \forall\ T\epsilon B(N)$ –(9)

Is an isometric isomorphism,we have to show that $|| J(T)|| = ||T||$and $|| J(T)|| = ||T^*|| = ||T||$ by (8)---(10)

We first claim that J is linear since if T, SϵB(N)andα, β are scalars $J(\alpha T+\beta S)=(\alpha T+\beta S)^*$ BY (9). Now $[(\alpha T+\beta S)^*(f)](x) =f[(\alpha T+\beta S)(x)]$ by (2)$=[\{\alpha T^*(f)] +\beta[S^*(f)\}](x) \Rightarrow[(\alpha T+\beta S)^*](f)=(\alpha T^*+\beta S^*)(f)= \alpha T^*+\beta S^* \Rightarrow J(\alpha T+\beta s) =$

$\alpha J(T)+\beta J(S) \Rightarrow$J is linear. Again we claim that J is one –one ,since $J(T) =J(S)$ $\Rightarrow T^* =S^*\Rightarrow|| T^*-S^*|| =0\Rightarrow|| T-S||^*=0$, by setting $\alpha =1,\beta=1\Rightarrow|| T-S||=0\Rightarrow$

T=S, i.e. J is one-one. We now claim that J preserves norm, since J(T)=T* ⇒ | ||J(T)|| = ||T*|| = ||T|| by(8) ,hence J(T)= ||T||⇒J preserves norm.

Consequently J: B(N) → B(N*) being linear ,one-one and norm preserving is an isometric isomorphism.Further,we claim that J reverses products ,since by J(T)=T* we have J(TS)=(TS)* and (TS)* (f) = f(TS) -------------------(12) ⇒[(TS)* (f)](x) = f[(TS)] (x) =[T* (f)] S(x) =F[S(x)] on setting F =T*(f)= {S*[T*(f)] } (x)= [(S*T*)(f)](x)-----------------------(13) ⇒(TS)* =S*T*⇒J(TS) =S*T* by (12) ⇒J reverses the products, Lastly we claim J preserves the identity transformation since if Ibe the identity transformation such that J(I) =I*,then[I* (f)] (x) = f [I(x)] by (2) = f(x)=[I(f)] (x) so [I* (f)] (x)=[I(f)] (x) ⇒ I*=I--------------------(14)ThusJ(I)=I*=I⇒Jpreservestheidentity transformation

Cor3 : Tis invertible ⇔T* is invertible

Proof: Here Tis invertible ⇔$TT^{-1} = T^{-1}T = I$ ⇔$(TT^{-1})^* = (T^{-1}T)^* = I^* = I$⇔ $(T^{-1})^*T^* = T^*(T^{-1})^* = I$⇔T* is invertible and $(T^*)^{-1} = (T^{-1})^*$