The Electromagnetic field from Lienard Wiechert Potential of Moving Point charge

Electromagnetic field $\vec{E}$ and $\vec{B}$ given by

$$\vec{B} = \text{Curl}\,\vec{A} \quad -(1) \qquad \text{and} \qquad E = -\text{grad}\,\phi - \frac{\partial A}{\partial t} \quad -(2)$$

Now from Lienard-Wiechert Potential, given by

$$\phi(r,t) = \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{\delta(t'-t+\frac{R}{c})\,dt'}{R(r,r')} \qquad (3)$$

$$\vec{A}(r,t) = -\frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\frac{V(t')\,\delta(t'-t+\frac{R}{c})\,dt'}{R(r,r')} \qquad (4)$$

Putting these value from (3) and (4) in eq (2) we get

$$\vec{E}(r,t) = -\left[\frac{q}{4\pi\epsilon_0}\nabla\int_{-\infty}^{+\infty}\frac{\delta(t'-t+\frac{R}{c})\,dt'}{R(r,r')} + \frac{\mu_0 q}{4\pi}\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\frac{v(t')\,\delta(t'-t+\frac{R}{c})\,dt'}{R(r,r')}\right] \qquad (5)$$

Now we note that the only dependence on the spatial coordinate $\vec{r}$ of the field point is through R. So gradient operation is equivalent to

$$\text{grad} = \vec{\nabla} \to \hat{n}\frac{\partial}{\partial R} \qquad -(6)$$

Here $\hat{n}$ is unit vector along $\vec{R} = (r-r')$.

So eq (5) will be

$$\vec{E}(\vec{r},t) = -\left[\frac{q}{4\pi\epsilon_0}\hat{n}\int_{-\infty}^{+\infty}\frac{\delta(t'-t+\frac{R}{c})\,dt}{R(r,r')} + \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\frac{v(t')\,\delta(t'-t+\frac{R}{c})\,dt'}{R(r,r')}\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\left[\frac{\hat{n}}{R^2}\delta(t'-t+\frac{R}{c}) - \frac{\hat{n}}{Rc}\delta'(t'-t+\frac{R}{c})\right]dt' + \frac{\mu_0}{4\pi}\int_{-\infty}^{+\infty}\frac{v(t')\,\delta'(t'-t+\frac{R}{c})\,dt'}{R(r,r')}$$

The prime on the delta function means differentiation with respect to its argument. Now using $\mu_0\epsilon_0 = \frac{1}{c^2}$ and hence substituting $\mu_0 = \frac{1}{c^2\epsilon_0}$ and $\vec{\beta} = \frac{\vec{V}}{c}$, we get

$$E(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\left[\frac{n}{R^2}\delta(t'-t+\frac{R}{c}) + \left(\frac{\vec{\beta}-n}{cR}\right)\delta'(t'-t+\frac{R}{c})\right]dt' \qquad (7)$$

Similarly substituting the value of A from (4) in eq (1) we get

$$B(r,t) = \frac{\mu_0 q}{4\pi}\nabla\times\int_{-\infty}^{+\infty}\frac{V(t')\,\delta(t'-t+\frac{R}{c})\,dt'}{R(r,r')}$$

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$$\vec{B}(r,t) = \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial R}\times\frac{\{v(t')\,\delta(t'-t+\frac{R}{c})\}}{R(r,r')}\right]dt'$$

$$= \frac{\mu_0 q}{4\pi}\int_{-\infty}^{+\infty}\left[n\times v\left\{\frac{\delta(t'-t+\frac{R}{c})}{R^2}+\frac{\delta'(t'-t+\frac{R}{c})}{Rc}\right\}\right]dr'$$

$$= \frac{\mu_0 q c}{4\pi}\int_{-\infty}^{+\infty}\left[n\times\vec{\beta}\left\{\frac{\delta(t'-t+\frac{R}{c})}{R^2}+\frac{\delta'(t'-t+R/c)}{R/c}\right\}\right]dt' \quad -(8)$$

Since $t_1 = t' + \frac{R}{c}$

So $\frac{dt_1}{dt'} = 1 + \frac{1}{c}\frac{R}{dt'} = 1 = \frac{V.R}{cR}$ —(9) $\left[\text{Since } \frac{dR}{dt} = -\frac{\vec{V}.\vec{R}}{cR}\right]$

So $= 1 - \vec{\beta}.n = k$ $\quad\{$Here $\vec{\beta} = \frac{V}{c}$ and $k = 1-\vec{\beta}.\hat{n}\}$

$$dt' = \frac{dt'}{k} \quad -(10)$$

Substituting (9) and (10) in eq (7) and (8) we get

$$E(r,t) = \int_{-\infty}^{+\infty}\left[\frac{n}{kR^2}\delta(t_1-1) + \frac{(\vec{\beta}-\hat{n})}{kcR}\delta'(t_1-t)\right]dt_1 \quad -(11)$$

$$B(r,t) = \frac{\mu_0 c q}{4\pi}\int_{-\infty}^{+\infty}\left[(n\times\vec{\beta})\left\{\frac{-\delta(t_1-t)}{kR^2}+\frac{1}{kcR}\delta'(t_1-t)\right\}\right]dt_1 \quad -(12)$$

Integrating by parts on the derivative of delta function, we get

$$E(r,t) = \frac{q}{4\pi\epsilon_0}\left[\frac{n}{kR^2} - \frac{d}{dt_1}\left(\frac{\vec{\beta}-n}{kcR}\right)\right] \quad -(13)$$

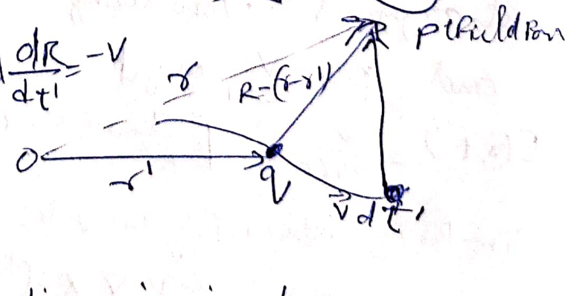
$$B(r,t) = \frac{\mu_0 c q}{4\pi}\left[\frac{n\times\vec{\beta}}{kR^2} - \frac{d}{dt_1}\left(\frac{n\times\vec{\beta}}{kcR}\right)\right] \quad -(14)$$

from (10), $dt_1 = k\,dt'$ so eq (13) and (14)

$$E(r,t) = \frac{q}{4\pi\epsilon_0}\left[\frac{n}{kR^2} + \frac{1}{ck}\frac{d}{dt'}\left(\frac{n-\beta}{kR}\right)\right] \quad -(15)$$

$$B(r,t) = \frac{\mu_0 c q}{4\pi}\left[\frac{\vec{\beta}\times n}{kR^2} + \frac{1}{ck}\frac{d}{dt'}\left(\frac{\vec{\beta}\times n}{kR}\right)\right] \quad -(16)$$

from fig we have $\frac{dR}{dt'} = -v.n$ and $\frac{dR}{dt'} = -V$ —(17)



$R = nR$

$$\frac{dR}{dt'} = n\frac{dR}{dt'} + R\frac{dn}{dt'}$$

rate of change of unit vector $\hat{n}$ with time is given by

$$\frac{d\hat{n}}{dt'} = -\frac{V}{R} + \frac{n(v.n)}{R} = -\frac{\beta c}{R} + \frac{n(\vec{\beta}c.n)}{R} = \frac{c}{R}\{\hat{n}(\vec{\beta}.\hat{n}) - (\hat{n}.\hat{n})\vec{\beta}\}$$

$$\frac{1}{c}\frac{d\hat{n}}{dt'} = \frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{R} \quad -(18)$$

So making this substitution, whenever it appears in (13), (15) and (16)

$$E(r,t) = \frac{q}{4\pi\epsilon_0}\left[\frac{n}{k^2R^2} + \frac{n}{ck}\frac{d}{dt'}\left(\frac{1}{kR}\right) - \frac{\vec{\beta}}{k^2R^2} - \frac{1}{ck}\frac{d}{dt'}\left(\frac{\vec{\beta}}{kR}\right)\right] \quad (19)$$

$$B(r,t) = \frac{\mu_0 cq}{4\pi}\left[\left\{\frac{\vec{\beta}}{k^2R^2} + \frac{1}{ck}\frac{d}{dt'}\left(\frac{\vec{\beta}}{kR}\right)\right\}\times\hat{n}\right] \quad (20)$$

from eq (19) and (20) magnetic Induction is related simply to the electric field by relation

$$\vec{B} = \frac{\hat{n}\times\vec{E}}{c} \quad (21)$$

The remaining derivatives needed in eq (19) and (20) are

$$\frac{d\vec{\beta}}{dt} = \dot{\vec{\beta}} \quad (22)$$

and

$$\frac{1}{c}\frac{d}{dt'}(kR) = \frac{R}{c}\frac{dk}{dt'} + \frac{k}{c}\frac{dR}{dt'} = \frac{R}{c}\frac{d}{dt'}(1-\hat{n}.\vec{\beta}) + (1-\hat{n}.\vec{\beta})(-\hat{n}.\vec{\beta})$$

$$\left[\text{Since } \frac{1}{c}\frac{dR}{dt'} = -\hat{n}.\vec{\beta}\right]$$

$$= \frac{R}{c}\left\{\frac{dn}{dt'}.\vec{\beta} - n.\frac{d\vec{\beta}}{dt'}\right\} - \hat{n}.\vec{\beta} + (\hat{n}.\vec{\beta})^2$$

$$= -\{\hat{n}\times(\hat{n}\times\vec{\beta})\}.\vec{\beta} - \frac{(n.\dot{\vec{\beta}})R}{c} - \hat{n}.\vec{\beta} + (\hat{n}.\vec{\beta})^2$$

$$= -\{(\hat{n}.\vec{\beta})\hat{n} - (\hat{n}.\hat{n})\vec{\beta}\}.\vec{\beta} - \frac{(\hat{n}.\dot{\vec{\beta}})R}{c} - \hat{n}.\vec{\beta} + (n.\vec{\beta})^2$$

$$= \beta^2 - n.\vec{\beta} - \frac{R}{c}(n.\dot{\vec{\beta}}) \quad (23)$$

Then eq (14) takes the form

$$E(r,t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{n}}{k^2R^2} + \frac{\hat{n}}{ck}\left(-\frac{1}{k^2R^2}\right)\frac{d}{dt'}(kR) - \frac{\vec{\beta}}{ck^2R} - \frac{\dot{\vec{\beta}}}{ck^2R} - \frac{\vec{\beta}}{ck}\left(-\frac{1}{k^2R^2}\right)\frac{d}{dt'}(kR)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{n}-\vec{\beta})}{k^2R^2} - \frac{(\hat{n}-\vec{\beta})}{ck^3R^2}\frac{d}{dt'}(kR) - \frac{\dot{\vec{\beta}}}{ck^2R}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{n}-\vec{\beta}}{k^2R^2} - \frac{(\hat{n}-\vec{\beta})}{k^3R^2}\left\{\beta^2 - \hat{n}.\vec{\beta} - \frac{R}{c}(n.\dot{\vec{\beta}})\right\} - \frac{\dot{\vec{\beta}}}{ck^2R}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{n}-\vec{\beta})}{k^3R^2}\left\{k - \beta^2 + \hat{n}.\vec{\beta} + \frac{R}{c}(\hat{n}.\dot{\vec{\beta}})\right\} - \frac{\dot{\vec{\beta}}}{ck^2R}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{n}-\vec{\beta}}{k^3R^2}\left\{(1-\hat{n}.\vec{\beta}) - \beta^2 + \hat{n}.\vec{\beta} + \frac{R}{c}\hat{n}.\dot{\vec{\beta}}\right\} - \frac{\dot{\vec{\beta}}}{ck^2R}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{k^3R^2} + \frac{\hat{n}-\vec{\beta}}{k^2Rc}\{(\hat{n}.\dot{\vec{\beta}}) - k\dot{\vec{\beta}}\}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{k^3R^2} + \frac{(\hat{n}-\vec{\beta})}{k^3Rc}\{(\hat{n}.\dot{\vec{\beta}}) - (1-\hat{n}.\vec{\beta})\dot{\vec{\beta}}\}\right] \quad (24)$$

using

$$\hat{n}\times(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}} = (\hat{n}.\dot{\vec{\beta}})(\hat{n}-\vec{\beta}) - \{\hat{n}.(\hat{n}-\vec{\beta})\}\dot{\vec{\beta}} = (\hat{n}.\dot{\vec{\beta}})(\hat{n}-\vec{\beta}) - (1-\hat{n}.\vec{\beta})\dot{\vec{\beta}}$$

we may write equation (24) as

$$E(r,t) = \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{k^3R^2} + \frac{1}{k^3Rc}\{\hat{n}\times(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right] \quad (25)$$

Now using $n = \frac{\vec{R}}{R}$ and $k = 1-\hat{n}.\vec{\beta} = 1-\frac{\vec{R}.\vec{\beta}}{R}$, i.e $kR = R-\vec{R}.\vec{\beta}$

then eq (25) will be

$$E(r,t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\vec{R}-(R\vec{\beta})(1-\beta^2)}{R-\vec{R}.\vec{\beta}} + \frac{\vec{R}\times(\vec{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c(\vec{R}-\vec{R}.\vec{\beta})^3}\right]_{\text{retaracted}} \quad (26)$$

and magnetic field induction is given by

$$\vec{B} = \frac{q}{4\pi\epsilon_0 Rc}\left[\frac{\hat{n}\times(\vec{R}-R\vec{\beta})(1-\beta^2)}{(R-\vec{R}.\vec{\beta})^3} + \frac{\hat{n}\times\{\vec{R}\times(\vec{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c(R-\vec{R}.\vec{\beta})^2}\right]$$

using $\mu_0\epsilon_0 = \frac{1}{c^2}$ and $R\times R = 0$ we may written as

$$B(r,t) = \frac{\mu_0\epsilon_0 q}{4\pi}\left[\frac{(\vec{\beta}\times\vec{R})(1-\beta^2)}{(R-\vec{R}.\vec{\beta})^3} + \frac{\hat{n}\times\{\vec{R}\times(\vec{R}-R\vec{\beta})\times\dot{\vec{\beta}}\}}{cR(R-\vec{R}.\vec{\beta})^3}\right]_{\text{retarded}} \quad (27)$$

Equation (26) and (27) represent required electromagnetic field due to a moving point charge