Radiation from an Accelerated charge at low velocity - [Larmor's formula] :→

The electromagnetic fields of a point charge in motion are (eqns)

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{(\vec{R}-R\vec{\beta})(1-\beta^2)}{(R-\vec{R}\cdot\vec{\beta})^3} + \frac{\vec{R}\times(\vec{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c(R-\vec{R}\cdot\vec{\beta})^3}\right] \quad -① $$

and

$$\vec{B}(\vec{r},t) = \left[\frac{\hat{n}\times\vec{E}}{c}\right] = \left[\frac{\vec{R}\times\vec{E}}{Rc}\right] \quad \cdots ②$$

in equation ① the first term in square braket represent velocity field, while second term represents acceleration field. Here (in this) The velocity fields are essential static fields and they are incapable of radiating energy. So for radiation we shall consider only acceleration fields given as

$$\vec{E}_a(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\vec{R}\times(\vec{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c[R-\vec{R}\cdot\vec{\beta}]^3}\right] \quad -③$$

with

$$\vec{B}_a(\vec{r},t) = \left[\frac{\hat{n}\times\vec{E}_a}{c}\right] = \left[\frac{\vec{R}\times\vec{E}_a}{Rc}\right] \quad -④$$

Where square bracket notation reminds us that all quantities in square bracket must be evaluated at retarded time. If the velocity of charge is small compared with the velocity of light, then -

$$\beta = \frac{v}{c} << 1 \quad \text{so that} \quad \vec{R}-R\vec{\beta} \to \vec{R}$$

Then the accelerated field equation ③ takes the form

$$\vec{E}_a(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\vec{R}\times(\vec{R}\times\dot{\vec{\beta}})}{R^3}\right] \cdots (3a) = \frac{q}{4\pi\epsilon_0 c}\frac{(\vec{R}\cdot\dot{\vec{\beta}})\vec{R}-(\vec{R}\cdot\vec{R})\dot{\vec{\beta}}}{R^3}$$

$$= \frac{q}{4\pi\epsilon_0 c}\left[\frac{(\vec{R}\cdot\dot{\vec{\beta}})\vec{R}-(R^2\dot{\vec{\beta}})}{R^3}\right] \quad \cdots\cdots (3b)$$

Using $\hat{n} = \frac{\vec{R}}{R}$ where $\hat{n}$ is unit vector along $\vec{R}$, we get -

$$\vec{E}_a(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c}\left[\frac{(\hat{n}\cdot\dot{\vec{\beta}}\,\hat{n}-\dot{\vec{\beta}})}{R}\right] \quad -④$$

If $\vec{a}$ is acceleration of the particle then $\vec{a} = \dot{\vec{v}} = c\dot{\vec{\beta}}$ or $\dot{\vec{\beta}} = \frac{\vec{a}}{c}$ and we get -

$$\vec{E}_a(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c}\left[\frac{(\hat{n}\cdot\vec{a})\hat{n}-\vec{a}}{R}\right] \quad ⑤$$

Now

$$\vec{B}_a = \left[\frac{\hat{n}\times\vec{E}_a}{\mu_0 c}\right] \quad ⑥$$

Therefore the instantaneous energy flux given by Poynting vector is

$$\vec{S}_a = \vec{E}_a \times \vec{H}_a = \frac{\vec{E}_a \times \vec{B}_a}{\mu_0}$$

[* All values on R.H.S correspond to retarded position of charges]

$$= \left[\frac{\vec{E}_a \times (\hat{n} \times \vec{E}_a)}{\mu_0 c}\right] = \frac{1}{\mu_0 c}\left[(\vec{E}_a \cdot \vec{E}_a) - (\vec{E}_a \cdot \hat{n})\vec{E}_a\right]$$

But $\vec{E}_a$ is perpendicular to $\vec{R}$ or $\hat{n}$, so $\vec{E}_a \cdot \hat{n} = 0$, hence we get

$$S_a = \frac{1}{\mu_0 c}|E_a|^2 \hat{n}$$

Putting the value of E_a from eq (5) we get –

$$S_a = \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c^2}\right)^2 \left|\left\{\frac{(\hat{n}\cdot\vec{a})\,\hat{n} - \vec{a}}{R}\right\}^2\right| \hat{n}$$

$$= \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c^2}\right)^2 \left[\frac{(\hat{n}\cdot\vec{a})^2 - 2(\hat{n}\cdot\vec{a})(\vec{a}\cdot\hat{n}) + a^2}{R^2}\right]$$

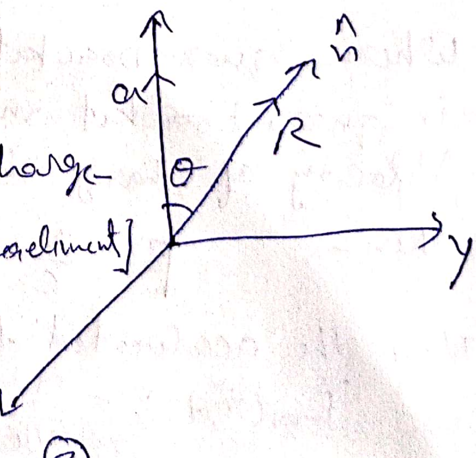
using $\mu_0\epsilon_0 = \frac{1}{c^2}$ and $\hat{n}\cdot\vec{a} = \vec{a}\cdot\hat{n}$ we obtain

$$S_a = \frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{4\pi c^3}\left[\frac{a^2 - (\hat{n}\cdot\vec{a})^2}{R^2}\right]\hat{n}$$

If θ is angle between $\hat{n}$ and $\vec{a}$ then $\hat{n}\cdot\vec{a} = a\cos\theta$, so we get

$$S_a = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4\pi c^3}\left[\frac{a^2 - a^2\cos^2\theta}{R^2}\right] = \frac{1}{4\pi\epsilon_0}\frac{q^2 a^2}{4\pi c^3 R^2}\sin^2\theta\,\hat{n} \quad \text{—(6)}$$

Eq (6) shows the power flow per unit area.



for Total Power radiation integrated eq (6) over a closed surface surrounding the charge

$$P = \int_S S_a \cdot \hat{n}\,da \quad \text{—(7)}$$

[Here da is an area element]

$$= \int_S \frac{1}{4\pi\epsilon_0}\frac{q^2 a^2}{4\pi c^3 R^2}\sin^2\theta\,\hat{n}\cdot\hat{n}\,da$$

[But $\hat{n}\cdot\hat{n} = 1$]

and $da = (R\,d\theta)(R\sin\theta\,d\phi)$

So. $$P = \int_0^{\pi}\int_0^{2\pi} \frac{1}{4\pi\epsilon_0}\frac{q^2 a^2 \sin^2\theta}{4\pi c^3 R^2}(R\,d\theta)(R\sin\theta\,d\phi) \quad \text{—(8)}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2 a^2}{4\pi c^3}\int_0^{\pi}\sin^3\theta\,d\theta \int_0^{2\pi} d\phi$$

$$\left[\text{But } \int_0^{2\pi} d\phi = 2\pi \text{ and } \int_0^{\pi}\sin^3\theta\,d\theta = \frac{4}{3}\right]$$

So $$P = \frac{1}{4\pi\epsilon_0}\frac{q^2 a^2}{4\pi c^3}\left(\frac{4}{3}\right)\cdot(2\pi) = \frac{1}{4\pi\epsilon_0}\frac{2q^2 a^2}{3c^3} \quad \text{—(9)}$$

This is Larmor's formula for power radiated by a slowly moving (nonrelativistic) accelerated charge.

Radiation from an Accelerated charge at high velocity :-

or Relativistic Generalisation of Larmor's formula -

we know Larmor's non relativistic formula is -

$$P = \frac{1}{4\pi\epsilon_0} \frac{2q^2 a^2}{3c^3} \quad \cdots\cdots \quad (1)$$

we know $\vec{a} = \frac{\vec{F}}{m} = \frac{d\vec{P}/dt}{m}$, then eq (1) will be

$$P = \frac{1}{4\pi\epsilon_0} \frac{2q^2 (\vec{a}.\vec{a})}{3c^2} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{m^2c^2} \left(\frac{d\vec{P}}{dt}.\frac{d\vec{P}}{dt}\right) - (2)$$

Here m is mass of the charge and $\vec{P}$ is momentum. According to Lorentz invariant generalisation eq (2)

$$P = \frac{1}{4\pi\epsilon_0} \cdot \frac{2}{3} \frac{q^2}{m^2c^2} \left(\frac{dP_\mu}{d\tau} \frac{dP_\mu}{d\tau}\right) - (3)$$

Here P_μ is the charge particles momentum energy four vector given by

$$P_\mu = \left[\vec{P}, \frac{iE}{c}\right] - (4)$$

and $d\tau$ is the proper time interval given by -

$$dt = \frac{d\tau}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{d\tau}{\sqrt{(1-\beta^2)}} = \gamma d\tau$$

Here $\gamma = \frac{1}{\sqrt{(1-\beta^2)}}$ (5)

To check that eq (3) reduces properly to eq (2) as $\beta \to 0$, we evaluate the four-vector scalar product -

$$\frac{dP_\mu}{d\tau} \frac{dP_\mu}{d\tau} = \left(\frac{dP_1}{d\tau}\right)^2 + \left(\frac{dP_2}{d\tau}\right)^2 + \left(\frac{dP_3}{d\tau}\right)^2 + \left(\frac{dP_4}{d\tau}\right)^2$$

$$= \left(\frac{d\vec{P}}{d\tau}\right)^2 + \left\{\frac{d}{d\tau}\left(\frac{iE}{c}\right)\right\}^2 = \left(\frac{d\vec{P}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2$$

But $d\tau = \frac{dt}{\gamma}$, so

$$\frac{dP_\mu}{d\tau} \frac{dP_\mu}{d\tau} = \gamma^2 \left(\frac{d\vec{P}}{dt}\right)^2 - \frac{\gamma^2}{c^2}\left(\frac{dE}{dt}\right)^2$$

Making this substitution in eq (3) we get -

$$P = \frac{1}{4\pi\epsilon_0} \cdot \frac{2}{3} \frac{q^2\gamma^2}{m^2c^3} \left[\left(\frac{dP}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right] - (6)$$

Now in order to express above equation in term of velocity and acceleration, let us use relativistic formulae

$$E = \frac{mc^2}{\sqrt{(1-\beta^2)}} \quad \text{and} \quad \vec{p} = \frac{m\vec{v}}{\sqrt{(1-\beta^2)}} \qquad -(7)$$

where m is the rest mass.
From (7) we have

$$\frac{dE}{dt} = \frac{d}{dt}\{mc^2(1-\beta^2)^{-1/2}\} = \frac{mc^2(\vec{\beta}.\dot{\vec{\beta}})}{(1-\beta^2)^{3/2}} = mc^2\gamma^3\,\vec{\beta}.\dot{\vec{\beta}} \qquad -(8)$$

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left\{\frac{m\vec{v}}{\sqrt{(1-\beta^2)}}\right\} = \frac{d}{dt}\left\{\frac{mc\vec{\beta}}{\sqrt{(1-\beta^2)}}\right\} = \frac{mc[\dot{\vec{\beta}}(1-\beta^2)+\vec{\beta}(\vec{\beta}.\dot{\vec{\beta}})]}{(1-\beta^2)^{3/2}}$$

$$\boxed{\frac{d\vec{p}}{dt} = mc\gamma^3[\dot{\vec{\beta}}.(1-\beta^2)+\vec{\beta}(\vec{\beta}.\dot{\vec{\beta}})]} \qquad (9)$$

using 8 and 9 in eq (6) becomes.

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^3}{m^2c^3}[m^2c^2\gamma^6\{\dot{\vec{\beta}}(1-\beta^2)+\vec{\beta}(\vec{\beta}.\dot{\vec{\beta}})\}-m^2c^2\gamma^6(\vec{\beta}.\dot{\vec{\beta}}^2)]$$

on simplification, we get

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c}\gamma^6[(\dot{\vec{\beta}})^2-\beta^2\dot{\beta}^2+(\vec{\beta}.\dot{\vec{\beta}})^2]$$

But $(\vec{\beta}\times\dot{\vec{\beta}}) = (\vec{\beta}.\dot{\vec{\beta}}) - \beta^2\dot{\beta}^2$

so

$$\boxed{P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^6}{c}[\dot{\vec{\beta}}-(\vec{\beta}\times\dot{\vec{\beta}})^2]} \qquad -(10)$$

This is relativistic generalisation of Larmor's formula.

Some Cases → (1) If the motion of the particle is nonrelativistic then $\beta \to 0$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}} \to 1$, so eq (10) will be

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c}(\dot{\vec{\beta}})^2$$

But $\dot{\vec{\beta}} = \frac{\vec{a}}{c}$ so we get

$$\boxed{P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\vec{a}^2}{c^3}} \qquad -(11)$$

[well known nonrelativistic Larmor's formula

Case-2 → if Motion of Particle's velocity and acceleration are parallel then $|\vec{\beta}\times\dot{\vec{\beta}}| = \beta\dot{\beta}\sin 0 = 0$ so that eq (10) will be

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^6}{c}\dot{\vec{\beta}}^2 = \frac{1}{4\pi\epsilon_0}\frac{2q^2\gamma^6 a^2}{3c^2} \qquad -(12)$$

This is Bremsstrahlung radiation,
This is also Synchrotron radiation

Case (3) if the motion of the particle is such that its velocity and acceleration are perpendicular; then

$$|\vec{\beta} \times \dot{\vec{\beta}}| = \beta \dot{\beta} \sin 90 = \beta \dot{\beta}$$

then eq (10)

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2 \gamma^6}{c} (\dot{\beta}^2 - \beta^2 \dot{\beta}^2) = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2 \gamma^6}{c} \dot{\beta}^2 (1-\beta^2)$$

$$\boxed{P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2 \gamma^4}{c} \dot{\beta}^2 = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2 \gamma^4 a^2}{c^3}} \quad (13)$$

This type of radiation occurs in circular acceleration like synchrotron or cyclotron where energy loss is drastic

Angular distribution of Radiation ~~Emitted~~ of Power radiated

For relativistic motion of charged particle the acceleration field depends on the velocity as well as acceleration and given as.

$$E_a = \frac{q}{4\pi\epsilon_0} \left[\frac{\vec{R} \times (\vec{R} - R\vec{\beta}) \times \dot{\vec{\beta}}}{c (R - \vec{R}\cdot\vec{\beta})^3} \right]$$

and

$$B_a = \frac{\hat{n} \times \vec{E}_a}{c} = \frac{\vec{R} \times \vec{E}_a}{Rc} \quad (1)$$

The Poynting vector is

$$[\vec{S}_a] = [E_a \times H_a] = \left[\frac{E_a \times B_a}{\mu_0}\right] = \left[\frac{\vec{E}_a \times (\hat{n} \times \vec{E}_a)}{\mu_0 c}\right] = \left[\frac{(\vec{E}_a \cdot \vec{E}_a)\hat{n} - (E_a \cdot \hat{n}) E_a}{\mu_0 c}\right]$$

$$= \left[\frac{E_a^2 \hat{n}}{\mu_0 c}\right] \qquad [\sin a \; \hat{E}_a \cdot \hat{n} = \vec{E}_a \cos 90° = 0] \quad (2a)$$

Radial Component of Poynting vector is

$$[\vec{S}_a \cdot \hat{n}] = \left[\frac{E_a^2}{\mu_0 c}\right] \quad (2b)$$

Here $(\vec{S}_a \cdot \hat{n})$ is the energy emitted per unit area per unit time detected at an observation point at time t due to radiation emitted by the charge at time $t' = t - R(t')/c$. If we want to calculate the energy radiated during a finite period, from $t' = T_1$ to $t' = T_2$ then we will write –

$$W = \int_{t = T_1 + [\{R(T_1)\}/c]}^{t = T_2 + [\{R(T_2)\}/c]} (\vec{S}_a \cdot \hat{n})\, dt = \int_{t' = T_1}^{t' = T_2} (\vec{S}_a \cdot \hat{n}) \frac{dt}{dt'} d\tau'$$

Here $(\vec{S}_q\cdot\hat{n})\left(\frac{dt}{dt'}\right)$ represent the power radiated per unit area is in terms of charge's own time.

The power radiated per unit solid angle is defined as

$$\frac{dP(t')}{d\Omega} = (\text{Power radiated per unit area}) \times \frac{4\pi R^2}{4\pi}$$
$$= R^2 \times (\text{Power radiated per unit area}) = R^2(\vec{S}_q\cdot\hat{n})\frac{dt}{dt'} \quad -(3)$$

As $t' = t - \frac{R}{c}$,

$$\frac{dt'}{dt} = 1-\frac{1}{c}\frac{dR}{dt} = 1-\frac{1}{c}\frac{dR}{dt}\frac{dt'}{dt} = \left[1+\frac{1}{c}\frac{dR}{dt'}\right]\frac{dt'}{dt} = 1$$

or $$\frac{dt'}{dt} = \frac{1}{1+\frac{1}{c}\frac{dR}{dt'}}$$

$$\frac{dR}{dt'} = -\frac{\vec{V}\cdot\vec{R}}{R} = -\vec{V}\cdot\hat{n} \quad \text{since } \hat{n} = \frac{\vec{R}}{R}$$

So $$\frac{dt'}{dt} = \frac{1}{1-\left(\frac{\vec{V}\cdot\hat{n}}{c}\right)} = \frac{1}{1-\hat{n}\cdot\vec{\beta}} =$$

$$\frac{dt}{dt} = 1-\hat{n}\cdot\vec{\beta} = k \quad -(4)$$

using eq (4) and (3) we have (5)

$$\frac{dP(t')}{d\Omega} = KR^2(\vec{S}_q\cdot\hat{n}) = KR^2\frac{E_q^2}{\mu_0 c} \quad [\text{using } 2b]$$

$$= \frac{KR^2}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0}\right)^2\left[\frac{\vec{R}\times(\vec{R}-R\vec{\beta})\times\dot{\vec{\beta}}}{c(R-\vec{R}\cdot\vec{\beta})^3}\right] = \frac{KR^2}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0}\right)^2\left[\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{c(1-\vec{n}\cdot\vec{\beta})^3}\right]^2$$

using $K = 1-\hat{n}\cdot\vec{\beta}$ and $\mu_0\epsilon_0 = \frac{1}{c^2}$ we get

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4\pi c}\frac{[\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}]^2}{c(1-\hat{n}\cdot\vec{\beta})^5} \quad -(6)$$

Eq (6) is general expression for the angular distribution of power radiated by an accelerated charge.