* Arrival Distribution Theorem:-

<u>Theorem</u>:- If the arrivals are completely random, then the probability distribution of numbers of arrivals in a fixed time interval follows a Poisson's Distribution

<u>Proof</u>:- In order to derive the arrival distribution in queues. we make the three assumptions

① There are 'n' units in the system at time 't' and the probability that exactly one arrival (birth) will occur during small time interval Δt be given by $\lambda(\Delta t) + O(\Delta t)$, where λ is the arrival rate independent of 't', and $O(\Delta t)$ includes the terms of high order of Δt.

② Secondly, assume that the time 'Δt' is so small that the probability of more than one arrival in time 'Δt' is $O(\Delta t)^2$ i.e. almost zero.

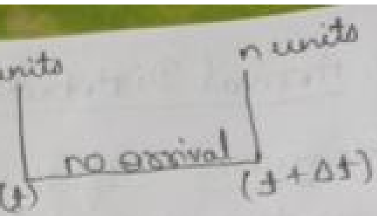
③ The number of arrivals in non-overlapping interval are statistically independent.

- The probability of 'n' arrivals in a time interval of 't' denoted by $P_n(t)$.

<u>Case I</u>:- <u>When $n>0$</u> :-

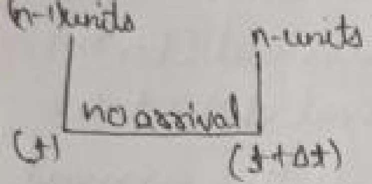
(i) There are 'n' units in the system at time 't' and no arrival takes place arriving time interval Δt. Hence, there will be 'n' units at time $(t+\Delta t)$ also probability of two combined events :

⇒ Probability of n-units at time t × Prob. of no arrival during Δt



$$= P_n(t)\,(1-\lambda\Delta t) \quad \text{—} \; (1)$$

(ii) Secondly, there are $(n-1)$ units in the system at time t, and one arrival takes place during Δt. Hence, there will remain 'n' units in the system at time $(t+\Delta t)$.



Prob. = Prob. of $(n-1)$ units at time 't' × Prob. of one arrival in time (Δt)

$$= P_{n-1}(t) \times \lambda(\Delta t) \quad \text{—} \; (2)$$

Adding (1) & (2), we get prob. of n arrivals at time $(t+\Delta t)$ is:-

$$P_n(t+\Delta t) = P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t \quad \text{—} \; (1)$$

Case II :- When $n=0$,

$P_0(t+\Delta t)$ = Prob. (no unit at time t) × Prob. (no arrival in time Δt).

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) \quad \text{—} \; (2)$$

Rewritting the eq^n (1) & (2) after taking terms –

$$P_n(t+\Delta t) - P_n(t) = P_n(t)(-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t \quad (n>0)$$

$$P_0(t+\Delta t) - P_0(t) = P_0(t)(-\lambda\Delta t) \quad (n=0).$$

Dividing both sides of above equation by Δt & then taking limit as $(\Delta t \to 0)$.

$$\lim_{\Delta t\to 0}\left[\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t}\right]=\frac{\Delta t\left[-\lambda P_n(t)+\lambda P_{n-1}(t)\right]}{\Delta t}$$

$$P_n'(t)=-\lambda P_n(t)+\lambda P_{n-1}(t) \quad \text{—— (3)}$$

- By definition of I^{st} derivative -

$$\lim_{\Delta t\to 0}\left[\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t}\right]=\frac{d}{dt}P_n(t)=P_n'(t).$$

$$\lim_{\Delta t\to 0}\left[\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t}\right]=\frac{-\lambda P_0(t)\cdot\Delta t}{\Delta t}$$

$$\Rightarrow P_0'(t)=-\lambda P_0(t) \quad \text{—— (4)}$$

By definition of I^{st} derivative we can write,

$$\lim_{\Delta t\to 0}\left[\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t}\right]=P_0'(t).$$

Rewritting it again eqn (3) & (4).

$$P_0'(t)=-\lambda P_0(t) \quad , n=0 \quad \text{—— (5)}$$

$$P_n'(t)=-\lambda(P_n(t))+\lambda P_{n-1}(t) \quad ; n>0 \quad \text{—— (6)}$$

This is Known as the System of differential-difference Eqn.

Now, to solve eqn (5) & (6).

$$(5) \Rightarrow \frac{P_0'(t)}{P_0(t)}=-\lambda$$

We can write it as; $\frac{d}{dt}\left[\log P_0(t)\right]=-\lambda$ —— (7)

Integrating both sides; of eqn (7) -

$$\Rightarrow \log P_0(t) = -\lambda t + A \quad \text{—— (8)}$$

using boundary condition, to obtain constant of integration -

$$\Rightarrow P_n(0) = \begin{cases} 1 & , \text{ for } n=0 \\ 0 & , \text{ for } n>0 \end{cases}$$

Putting $t=0$ in (8).

$\Rightarrow \log 1 = -\lambda(0) + A \Rightarrow A = 0$, then using eqn (8)

$$\Rightarrow \log P_0(t) = -\lambda t$$

$$\Rightarrow P_0(t) = e^{-\lambda t} \quad \text{—— (9)}.$$

Putting $n=1$, in eqn (6),

$$\Rightarrow P_1'(t) = -\lambda P_1(t) + \lambda P_0(t)$$

$$\Rightarrow P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t} \quad \text{—— (10)}$$, using eqn (9)

As this is the linear differential eqn of first order it can be easily solved by multiplying both of eqn (10) by integrating factor -

Eqn (10)

$$\Rightarrow P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}.$$

$$\Rightarrow e^{\lambda t}\left[P_1'(t) + \lambda P_1(t)\right] = \lambda$$

$$\Rightarrow e^{\lambda t} P_1'(t) + \lambda e^{\lambda t} \cdot P_1(t) = \lambda.$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda t} P_1(t)\right] = \lambda \quad \text{—— (11)}$$

Integrating both sides of eqn (11), we get -

$$\Rightarrow e^{\lambda t} P_1(t) = \lambda t + B \quad \text{--- (12)}$$

'B' is constant of integration, In order to determine B put ($t=0$) in eqn (12), we get

$$\Rightarrow e^0 P_1(0) = \lambda \cdot (0) + B \qquad \because P_1(0) = 0.$$

$$\Rightarrow \boxed{B = 0}$$

So, eqn (12) becomes,

$$\Rightarrow e^{\lambda t} P_1(t) = \lambda t$$

We can write it as;

$$\Rightarrow \boxed{P_1(t) = \frac{\lambda t e^{-\lambda t}}{1!}} \quad \text{--- (13)}$$

Put $n=2$ in eqn (6) & using eqn (13)

$$\Rightarrow P_2'(t) + \lambda P_2(t) = \lambda P_1(t).$$

$$\Rightarrow e^{-\lambda t} \cdot P_2'(t) + \lambda e^{\lambda t} P_2(t) = \frac{\lambda(\lambda t)}{1!}$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda t} P_2(t)\right] = \frac{\lambda(\lambda t)}{1!}$$

Integrating w.r.to 't' -

$$\Rightarrow e^{\lambda t} P_2(t) = \frac{\lambda^2 t}{2!} \cdot t + C$$

$$\Rightarrow e^{\lambda t} \cdot P_2(t) = \frac{(\lambda t)^2}{2!} + C$$

Put $t=0$, $P_2(0) = 0$ to obtain 'C' -

$\Rightarrow P_2(t) = \frac{(\lambda t)^2}{2!} e^{-\lambda t}$, for $n=2$ — (14)

Similarly, $P_3(t) = \frac{(\lambda t)^3}{3!} e^{-\lambda t}$, for $n=3$

⋮

for $n=m$,

$$\boxed{P_m(t) = \frac{(\lambda t)^m}{m!} e^{-\lambda t}} \quad \text{for } n=m \quad — (15)$$

It can be proved the result (15) is also true for $n=(m+1)$, then by induction hypothesis result eq^n (15) will be true for general values of n.

Put $n=(m+1)$ in eq^n (6), (using eq^n (15) also).

$$P'_{m+1}(t) + \lambda P_{m+1}(t) = \lambda \cdot \frac{(\lambda t)^m}{m!} e^{-\lambda t}$$

$$\therefore \frac{d}{dt}\left[e^{\lambda t} \cdot P_{m+1}(t)\right] = \frac{(\lambda t)^m \cdot (\lambda)}{m!}$$

Integrating,

$$e^{\lambda t} P_{m+1}(t) = \frac{(\lambda t)^{m+1}}{(m+1)\, m!} + D.$$

Put $t=0$, $P_{m+1}(0)=0$ & $D=0$.

$$\therefore P_{m+1}(t) = \frac{(\lambda t)^{m+1}}{(m+1)\, m!}$$

Hence, in general,

$$\Rightarrow \boxed{P_n(t) = \frac{(\lambda t)^n \cdot e^{-\lambda t}}{n!}}$$

This is a Poisson Distribution.

Hence, Proved