Module - 1

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(Riemann - Stieltjes Integral)

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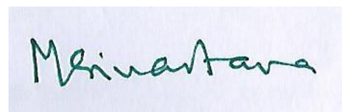

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THE RIEMANN - STIELTJES INTEGRAL

LECTURE -1

Today we will discuss about the Riemann Stieltjes Integral and some of its properties.

We know that the integral calculus is the outcome of an attempt to solve the problems of finding the area of the plane bounded by curves, in this process it is necessary to divide area into a very large number of small elements and then to obtain the limit of the sum of all these elements when each is infinitesimally small and their number is indefinitely great. Afterwards it was seen that the process of integration could be viewed as a process inverse of differential.

Riemann was the first scholar to give a satisfactory, rigorous arithmetic definition of the integral of a bounded function and also established a necessary & sufficient condition for existence of the definite integral of function.

Definition of Riemann Integral : Consider a bounded real valued function f (x) defined on closed interval [a, b] = I .By a partition of I ,we mean a finite set of real number P = $\{x_0,x_1,x_2$-----$x_n\}$ such that $a=x_0 <x_1 <x_2$ ---,$x_n =b$ The closed interval $[x_0 ,x_1]$,$[x_1,x_2]$ ------------$[x_{n-1},x_n]$ constitute the segments of partition. we denote

the sub-interval $[x_{r-1},x_r]$ and its length $x_r – x_{r-1}$ by δ_r . the greatest of the lengths of sub interval is called norm denoted by II P II = Max.{ $\delta_{r,}$: r=1,2, --------n}.

As we have already studied about lower, upper bound, sup & inf. of function in B.Sc. 3_{rd} Year,so let m & M be the inf. & sup of bounded function f(x) of [a, b] respectively. Now form the sum

$s = L(P, f) = \sum^{n}_{r=1} m_r \delta_{r,}$ $S = U(P,f) = \sum^{n}_{r=1} M_r \delta_{r,}$ The sum s and S are called Darboux sums. They are also called the lower & upper Riemann sum respectively, evidently s ≤ S and ¯ r=1,2,3 -----n

We have **$m \le m_r \le M_r \le M$** therefore **$\sum^{n}_{r=1} m\, \delta_r \le \sum^{n}_{r=1} m_r\, \delta_r \le \sum^{n}_{r=1} M_r\, \delta_r \le \sum^{n}_{r=1} M\, \delta_r$**

i.e.**$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$** Now consider all possible partition of [a,b]. Now we define upper integral & lower integral as follows

$\overline{\int}_a^b f(x)\,dx$ = inf. {U(P,f) : P is a partition of [a ,b] } and $\underline{\int}^b_a f(x)\,dx$ =Sup{L(P, f):P is a partition of [a,b]}

If $\overline{\int}_a^b f(x)\,dx = \underline{\int}^b_a f(x)\,dx = \int^b_a f(x)\,dx$ then we say that **f is Riemann integrable** or **R-integrable or integrable over [a ,b] denoted by (R)[a,b], or simply $\int^b_a f(x)\,dx$**

Now we will study about Riemann–Stieltjes integral which is the generalized concept of Riemann Integral given by Thomas Joanes Stieltjes (1856 – 1894).

Definition of Riemann Stieltjes Integral : Consider a bounded real valued function f (x) defined on closed interval [a, b] = I .By a partition of I ,we mean a finite set of real number P = $\{x_0, x_1, x_2$-------- --$x_n\}$ such that $a=x_0 < x_1 < x_2$ ---,$x_n = b$.The closed interval $[x_0, x_1]$,$[x_1, x_2]$ -----------$[x_{n-1}, x_n]$ constitute the segments of the partition. we denote the sub-interval $[x_{r-1}, x_r]$ and its length $x_r - x_{r-1}$ by δ_r. the greatest of the lengths of sub interval is called norm denoted by II P II =

max{ δ_r, :: r=1,2, --------n}.

As we have already studied about lower, upper bound, sup & inf. of function ,so let **m & M be the inf. & sup of bounded function f(x) of [a, b] respectively**. Let

m_r = inf.{ f(x) : x ϵ $[x_{r-1}, x_r]$} and M_r = Sup.{ f(x) : x ϵ $[x_{r-1}, x_r]$} . Let **g be monotonically non decreasing function on [a, b] ,write $\delta g_r = g(x_r) - g(x_{r-1})$ .then $\delta g_r \geq 0$.**

Now form the sum

s = L (f,g,P) = $\sum^n_{r=1} m_r \delta g_r$ S = U (f,g,P) = $\sum^n_{r=1} M_r \delta g_r$, These sums s and S are respectively called lower & upper Riemann -Stieltjes sums .evidently s ≤ S and r=1,2,3 --n

We have **$m \leq m_r \leq M_r \leq M$** therefore **$\sum^n_{r=1} m_r \delta g_r \leq \sum^n_{r=1} m_r \delta g_r \leq \sum n_{r=1} M_r \delta g_r \leq$**

$\sum^n_{r=1}$ **M** δg_r i.e. **m(b – a) ≤ L(P,f,g) ≤ U(P) ≤ M (b –a)** Now consider all possible partition of [a, b] ,now we define upper & lower Riemann -Stieltjes integral as follows $\overline{\int}_a^b f(x)\, dg$ =inf. {U(P,f,g,) : P is a partition of [a ,b] } and $\underline{\int}^b_a f(x)\, dg$ = Sup{L(P, f,g,):P is a partition of [a,b]}

If $\overline{\int}_a^b f(x)\, dg = \underline{\int}^b_a f(x)\, dg = \int^b_a f(x)\, dg$ **then we say that the integral is Riemann-stieltjes Integral denoted by R(g) or R-S (g), or simply=**$\int^b_a f(x)\, dg$ **over [a,b],the function f is called the integrand & g is called integrator.**

Riemann -Stieltjes Integral as a limit of sums : Let f be a bounded and g be monotonically increasing function on [a,b].Let P = { a = x_0, x_1, x_2-------x_{n-1} x_n, =b}be a partition of [a,b] and let $t_i \in [x_{i-1}, x_i]$,then we define the R-S sums of f relative to g on [a,b] as **S(P,f,g) =** $\sum^n_{i=1} f(t_i)\, \delta g i$ .The **sum S(P,f,g)** is said to be convergent to a **limit ,as** $||P|| \to 0$ and in this case $\int^b_a fdg$ **= S(P,f,g) =** $\sum^n_{i=1} f(t_i)\, \delta g i$

Refinement of a partition : A partition P^* is said to be refinement of P if $P \subset P^*$. In this case we say that P^* is finer than P i.e. P^* contains at least one point more than P. If P^* is the common refinement of the partitions P_1 & P_2 then $P^* = P_1 \cup P_2$.

Theorem 1: The lower integral can not exceed upper integral.

Pr**oof :** let P_1 & P_2 be two partition of [a,b] i.e.$P_{1,}$ $P_2 \epsilon P[a, b]$ then $L[P_1,f,g] \leq L\ P^*,f,g]$ $\leq U[P^*,f\ ,g] \leq U[P_2,f,g]$, then we have $L\ [P_1,f,g] \leq U[P_2,f,g]$ **--------(1)** now if we kept P_2 fixed and take Sup. Overall P_1, then from equation (1) we have $_{-}\int f(x)\ dg \leq [U(P_2,f,g)]$ **. -------(2)** The theorem follows by taking the inf. over all P_2 in (2)by definition of Lower & upper integral.

Theorem 2 : If P^* is the common refinement of P then

1) $L[P,f,g] \leq L[P^*,f,g]$ ------------- (1)
2) $U[P^*,f,g] \leq U[P,f,g]$ ---------------(2)

Proof : To prove **(1)** suppose first that P^*contains just one point more than P. Let this extra point be y ,and suppose $x_{i-1} < y < x_i$,where x_{i-1} & x_i are two consecutive points of P. Let w_1= inf. f(x) in $(x_{i-1} \leq x \leq y)$ & w_2 = inf. f(x) in $(y \leq x \leq x_i)$ clearly w_1 & $w_2 \geq m_i$ because m_i = inf. f(x) in $(x_{i-1} \leq x \leq x_i)$, hence

$L[P^*,f, g] - L[P, f, g]$ $= w_1\ [g(y) - g(x_{i-1})]+ w_2[\ g\ (x_i) - g\ (y)] - m_i\ [g\ (x_i\ ) - g\ (x_{i-1})]$

$= (w_1 - m_i)[\ g\ (y) - g(x_{i-1})] +(\ w_2 - m_i)[g(x_i) - g(y)]$ **≥ 0** ,hence we have the result.(1)

To prove (2) suppose first that P^*contains just one point more than P. Let this extra point be y, and suppose $x_{i-1} < y < x_i$ where x_{i-1} & x_i are two consecutive points of P. Let u_1 = Sup. f(x) in $(x_{i-1} \leq x \leq y)$ & u_2 = Sup. f(x) in $(y \leq x \leq x_i)$ clearly

u_1 **&** $u_2 \le M_i$ **as** M_i = Sup. f(x) in $(x_{i-1} \le x \le x_i)$, hence

U[P*,f, g] - U[P, f, g] $= u_1 [g(y) - g(x_{i-1})] + u_2[g(x_i) - g(y)] - M_i [g(x_i) - g(x_{i-1})]$

$= (u_1 - M_i)[g(y) - g(x_{i-1})] + (u_2 - M_i)[g(x_i) - g(y)] \le 0$,hence we have the **result. (2)**

Theorem 3: let $f \in R(g)$ on [a ,b] if and only if $\forall\ \varepsilon > 0\ \exists$ a partition P of [a,b] such that $U[P, f, g] - L[P, f, g] < \varepsilon$ **-----------------(1)**

Proof: For every partition P we have $L[P, f,g] \le \underline{\int}_a^b f(x)dg \le \overline{\int}_a^b f(x)\, dg \le U[P,f,g]$,

thus (1) implies by using definition of R-S integral $0 \le \overline{\int}_a^b f(x)\, dg - \underline{\int}_a^b f(x) < \varepsilon$, hence if (1) can be satisfied for every $\varepsilon > 0$,we have $\overline{\int}_a^b f(x)\, dg = \underline{\int}_a^b f(x)\, dg$ i.e.

$f \in R(g)$.

Conversely ,suppose $f \in R(g)$ and let $\varepsilon > 0$,be given then there exists partition

P_1 &P_2 such that $U[P_2,f,g] - \int_a^b f(x)\, dg < \varepsilon/2$ **--------(2)** and

$$\int_a^b f(x)\, dg - L[P_1,f,g] < \varepsilon/2 \quad \text{----------}(3)$$

We choose P to be common refinement of P_1 & P_2.Then Th.2 together with equations (2) & (3) show that $U[P, f, g] \le U[P_2,f,g] < \int_a^b f(x)\, dg + \varepsilon/2 < L[P_1,f,g] + \varepsilon \le L[P, f, g] + \varepsilon$ **.-----(4)** thus from equation (4) we get the result $U[P,f, g] - L[P, f ,g] < \varepsilon$.

LECTURE -2

Now we will study some more properties& Theorems on Riemann –Stieltjes Integral

Theorem 4 : 1) If f is continuous and g be monotonic non- decreasing on [a,b] then $f \in R(g)$ on [a,b].

2) If $P = \{ a = x_0, x_1, x_2 \text{-------} x_{n-1}\, x_n = b\}$ and s_i ,t_i are arbitrary points in $[x_{i-1}, x_i]$then

$\sum^n_{r=1} | f(s_i) - f(t_i)| \delta g_i < \varepsilon$

3) Moreover , given $\varepsilon > 0$,there exist $\delta>0$ such that $|\sum^n_{r=1} f(t_r)\, \delta g_i - \int^b_a f(x)\, dg| < \varepsilon$

Proof : 1) Let $\varepsilon > 0$ be given. Choose $\eta > 0$ so that $[g(b) - g(a)]\, \eta < \varepsilon$, since f is uniformly continuous on [a, b],there exists a $\delta > 0$ such that

$|f(x) - f(t)| < \eta$ **------------ (1),** if $x, t \in [a, b]$, and $|x-t| < \delta$

If P is any partition of [a, b] then $M_i - m_i \leq \eta$ for (I =1,2 ----n) and therefore

$U[P,f,g] - L[P,f,g] = \sum^n_{i=1} (M_i - m_i)\delta g_i \leq \eta \sum^n_{i=1} \delta g_i = \eta[g(b)-g(a)] < \varepsilon$ hence $f \in R(g)$ by Th3.

2) Since s_i ,$t_i \in [x_{i-1}, x_i]$,hence $f(s_i)$ & $f(t_i)$ lie in $[m_i, M_i]$, so that

$\{|f(s_i) - f(t_i)|\}\delta g_i \leq M_i - m_i \Rightarrow \sum^n_{i=1}|f(s_i) - f(t_i)|\ \delta g_i \leq U[P,f,g] – L[P,f,g] \leq \varepsilon$.hence proved

3) : Since f is R-S Integrable relative to g ,we have

$$\overline{\int}_a^b f(x)\, dg = \underline{\int}^b_a f(x)\, dg = \int^b_a f(x)\, dg \quad \text{---------------------}\mathbf{(2)}$$

Now $\overline{\int}_a^b f(x)\, dg = \inf.[U(P,f,g) \Rightarrow \overline{\int}_a^b f(x)\, dg \leq [U(P,f,g)]$ and $\overline{\int}_a^b f(x)\, dg \leq [U(P,f,g)]$

$\underline{\int}^b_a f(x)\, dg = \text{Sup}\ [L(P, f, g)] \Rightarrow \underline{\int}^b_a f(x)\, dg \geq [L(P, f, g)]$ whence we get

$\overline{\int}_a^b f(x)\, dg + \varepsilon > [U(P,f,g)]$ and $\underline{\int}^b_a f(x)\, dg - \varepsilon < [L(P,f,g)]$ by (2) we have

$\int^b_a f(x)\, dg - \varepsilon < [L(P,f,g)] < U(P,f,g)] < \int_a^b f(x)\, dg + \varepsilon$ ----------------**(3)** ,

if $t_i \in [x_{i-1},x_i]$ be arbitrary then obviously $m_i \leq f(t_i) \leq M_i$ and so

$$L(P, f, g)] \leq \sum f(t_i)\, \delta g_i \leq U(P, f, g)] \quad \text{----------------------------}\ \mathbf{(4)}$$

Combining (3) & (4) we have $\int^b_a f(x)\, dg - \varepsilon \leq \sum f(t_i)\, \delta g_i \leq \int_a^b f(x)\, dg + \varepsilon$ or we have

$-\varepsilon < \int^b_a f(x)\, dg - \sum f(t_i)\, \delta g_i < \varepsilon$ ie. $|\sum^n_{r=1} f(t_r)\, \delta g_i - \int^b_a f(x)\, dg| < \varepsilon$ hence proved.

Theorem5 : Let f be monotonic and g be continuous and monotonic non decreasing on [a,b] then $f \in R(g)$

Proof : Let $\varepsilon > 0$ .since g is continuous on[a ,b] ,it takes all the values between g(a) & g(b) ,also g is monotonic non decreasing we can therefore choose a partition P of [a, b] such that $\delta g_r = g(x_r) – g(x_{r-1}) = \{g(b) – g(a)\}/n$, for $r = 1,2,\text{------}\ n$ let $m_r = \inf.(f)$ & $M_r = \text{Sup}\ (f)$, also let f be monotonic non-decreasing then $m_r = f(x_{r-1})$ & $M_r = f(x_r)$ Now

$U[P, f, g] – L[P, f, g] = \sum^n_{r=1} (M_r - m_r) \delta g_r = \sum^n_{r=1} \{f(x_r) - f(x_{r-1})\} \{g(b) – g(a)\} /n$

$= \{f(b) – f(a)\} \{g(b) – g(a)\} /n$ when n is sufficiently large then R.H.S.becomes arbitrary small then $U[P, f ,g] – L[P, f, g] < \varepsilon$ i.e. $f \in R (g)$

Theorem6 : If $f \in R (g)$ on [a ,b] then c.f ∈ R(g) on [a, b] where c is constant and

$$\int^b_a cf (x) dg = c\int^b_a f (x) dg$$

Proof : Let $f \in R (g)$ on [a ,b] then given ε>0 ,there exist partition P of [a, b] such that $U[P, f ,g] – L[P, f, g] < \varepsilon$ and $\overline{\int}_a^b f(x)\, dg = \underline{\int}^b_a f (x)\, dg = \int^b_a f (x)\, dg$ **--------- (1)**

and (cf)(x) =c f(x) and so $U[P, c.f ,g] =c\, U[P ,f ,g]$ & $L[P, cf ,g] = c\, L[P, f ,g]$ **--------(2)**

Thus $U[P, cf ,g]- L[P ,cf, g] =c \{U[P, f, g] – L[P ,f ,g]\} < c\, \varepsilon = \varepsilon_1$,hence c.f ∈ R(g) on [a, b] ie. $\overline{\int}_a^b cf(x)\, dg = \underline{\int}^b_a cf (x)\, dg = \int^b_a cf (x)\, dg$ **---------(3).**Taking infimum of both sides of (2) for all partition P of [a ,b] we get $\overline{\int}_a^b cf(x)\, dg = c\overline{\int}_a^b f(x)\, dg$ **----(4)**

therefore $\int^b_a cf (x)\, dg = c \int^b_a f (x)\, dg$,hence proved.

Theorem 7: If $f \in R (g)$ on [a,b] then so is |f| and $|\int^b_a f (x)\, dg| \le |\int^b_a |f (x)|\, dg$

Proof: Since $f \in R (g)$ on [a ,b] ,f is bounded and g(x) is monotonic non- decreasing on [a,b]. Also given ε > 0 there exists partition P such that $U[P, f ,g] – L[P, f ,g] < \varepsilon$

ie. $\sum^n_{i=1}(M_i – m_i) \delta g_i < \varepsilon$ **--------(1)** where $m_i = \inf(f)$ & $M_i = \text{Sup } (f)$ in $[x_{i-1},x_i]$, and

$\delta g_i = g(x_i) - g(x_{i-1})$. Let $M_i' = Sup(|f|)$ & $m_i' = inf(|f|)$ in $[x_{i-1}, x_i]$, if x ,y $\epsilon[x_{i-1}, x_i]$ then $|\{|f(x)| - |f(y)|\}| \leq |f(x) - f(y)$,this suggests that $M_i' - m_i' \leq M_i - m_i$ so we have $\sum^n_{i=1} M_i' - m_i' \delta g_i \leq \sum^n_{i=1} M_i - m_i \delta g_i < \varepsilon$ so that $U[P, |f|, g] - L[P, |f|, g] < \varepsilon$ therefore $|f| \epsilon R(g)$ Further $M_i \leq M_i'$ or, $| \sum^n_{i=1} M_i \delta g_i | \leq \sum^n_{i=1} M_i' \delta g_i$, making $|| P \to 0$,we get $|\int^b_a f(x)\, dg| \leq |\int^b_a |f(x)|\, dg$

LECTURE-3

Today we will study some special properties of R-S Integral

Theorem8 : Suppose $f\in R(g)$ on $[a, b]$, $m\leq f\leq M$, ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a, b]$, then $h\in R(g)$ on $[a, b]$.

Proof: Choose $\varepsilon > 0$. since ϕ is uniformly continuous on $[m, M]$, there exists $\delta>0$ such that $\delta<\varepsilon$ and $|\phi(s) - \phi(t)|<\varepsilon$ if $|s-t|\leq\delta$ **and $s, t\in[m, M]$. Since $f\in R(g)$ there is a partition P of [a, b] such that $U(P,f,g) - L(P,f,g) < \delta^2$ --------------------(1)** .

Let m_i & M_i be the inf & Sup of f on [a,b] and let m_i' and M_i' be the same for function h divide the numbers 1------n into two classes : $i\in A$ if $M_i - m_i<\delta$ and $I\in B$ if $M_i - m_i\geq\delta$. For $i\in A$ our choice of δ shows that $M_i' - m_i'\leq\varepsilon$. F or $i\in B$ $M_i' - m_i'\leq 2k$, where $K = \text{Sup}\,|\phi(t)|$, $m\leq t\leq M$. by (1) we have

$\delta\sum\delta g_i\leq\sum_{i\in B}(M_i - m_i)\,\delta g_i<\delta^2$, so that $\sum_{i\in B}\delta g_i<\delta$ it follows that **$U(P,h,g) - L(P,h,g)$**

$= \sum_{i\in A}(M_i' - m_i')\,\delta g_i+\sum_{i\in B}(M_i - m_i)\,\delta g_i\leq\varepsilon[g(b)- g(a)]+2K\delta<\varepsilon[g(b)- g(a)+2K]$,

since ε was arbitrary ,Th. 3 implies $h\in R(g)$.

Theorem9 If $f_1\in R(g)$ & $f_2\in R(g)$ then $f_1 + f_2\in R(g)$ and $\int_a^b f_1 + f_2\,dg = \int_a^b f_1dg + \int_a^b f_2dg$

Proof : If $f = f_1 + f_2$, and P is any partition of [a ,b] ,let m'_r, M'_r , M''_r, m''_r & m_r, M_r be the inf &Sup of f_1, f_2 & f then we have $m'_r\leq f_1\leq M'$, $m''_r\leq f_2\leq M''_r$ &

$m_r\leq f\leq M_r\Rightarrow m'_r + m''_r\leq f_1 + f_2\leq M'_r + M''_r$ and $m_r\leq f_1 + f_2\leq M_r\Rightarrow$

$m'_r + m''_r \le m_r \le M_r \le M'_r + M''_r$ ---------- (1) .now by (1) we have the inequality

$L(P,f_1,g] + L(P,f_2,g] \le L(P,f,g] \le U[P,f,g] \le U[P,f_1,g] + U[P,f_2,g]$ ----------------(2)

If $f_1 \in R(g)$ & $f_2 \in R(g)$ let $\varepsilon > 0$ be given ,there are partitions P_j (j =1,2)such

$[P_j, f_j, g] - L(P_j, f_j, g] < \varepsilon$ .in these inequalities if P_1 & P_2 are replaced by common refinement P then (2) implies $U[P,f\ g] - L(P,f, g] < 2\varepsilon$,since f_1 & $f_2 \in R(g)$ which proves that $f \in R(g)$

With partition P,we have $U[P,f_j ,g] <$ $\int_a^b f_j dg + \varepsilon$ (j= 1,2), then (2) implies that $\int_a^b f dg \le$ **$U[P,f,g] <$** $\int_a^b f_1 dg + \int_a^b f_2 dg + 2\varepsilon$ since ε is arbitrary we have

$\int_a^b f dg \le \int_a^b f_1 dg + \int_a^b f_2 dg$ **----------------(3)** ,if we replace **f_1 & f_2 by $- f_1$ &$- f_2$** the inequality (3) reversed i.e. $\int_a^b f dg \ge \int_a^b f_1 dg + \int_a^b f_2 dg$ **----------- ---(4)** hence by (3) & (4) we get the result.

Theorem10: (a) If $f \in R(g)$ on [a ,b] then $f^2 \in R(g)$ on [a ,b]

(b) If $f \in R(g)$ and $g \in R(g)$ then $fg \in R(g)$

proof : (a) since f is bounded on [a,b] hence ∃, $M > 0$ such that $|f(x)| \le M$, $\forall\ x \in [a ,b]$.Since $f \in R(g)$ and therefore for a given $\varepsilon > 0$ ∃ a partition P such that $U[P,f, g] - L[P,f,g] < \varepsilon / 2M$ **------ (1)** .Let M_r ,m_r and M'_r, m'_r be respectively the Sup and Inf of **f and f^2** in $[x_{r-1},x_r]$ then for any two points ζ_1 and $\zeta_2 \in [x_{r-1},x_r]$, wehave $|f^2(\zeta_1)-f^2(\zeta_2)|=|f(\zeta_1)+f(\zeta_2)|.|f(\zeta_1)-f(\zeta_2)|\le\{|f(\zeta_1)|+|f(\zeta_2)|\}.|f(\zeta_1)- f(\zeta_2)|\le$ $\{M+M.f(\zeta_1)-f(\zeta_2)|$.This suggests that $M'_r - m'_r \le 2M(M_r- m_r) \Rightarrow \sum_{r=1}^{n} (M'_r - m_r')\delta g_r$

$\leq 2M\sum^{n}_{i=r} (M_r - m_r)\delta x_r \Rightarrow U[P,f^2,g] - L[P,f^2,g] \leq 2M\{U[P,f,g]-L[P,f,g]\} \leq 2M\ (\varepsilon/2M)=\varepsilon$

by(1) $\Rightarrow U[P,f^2,g] - L[P,f^2,g] < \varepsilon$,hence $f^2 \in R\ (g)$.

(b)we take $\phi(t) = t^2$ Theorem 8. Shows that $f^2 \in R\ (g)$ if $f \in R\ (g)$.The identity

$4fg = (f + g)^2 - (f - \mathbf{g})^2$ completes the proof.

Theorem11 : If $f_1 \leq f_2$ then $\int^b_a f_1 dg \leq \int^b_a f_2 dg$

Proof : As $f_1 (x) \leq f_2(x)$,hence $f_2(x) - f_1(x) \geq 0$ on[a,b] and g is monotonically increasing in [a,b] so g(b) >g(a) and $\int^b_a (f_2 - f_1) dg \geq 0 \Rightarrow \int^b_a f_2 dg - \int^b_a f_1 dg \geq 0$ this implies $\int^b_a f_1 dg \leq \int^b_a f_2 dg$.

Theorem 12: If $f \in R(g)$ on [a ,b] and if $a < c < b$ then $f \in R(g)$ on [a, c]and [c, b] and

$$\int^b_a fdg = \int^c_a fdg + \int_c^b fdg.$$

Proof : Since $f \in R(g)$ on [a,b] ,given $\varepsilon > 0$ Ǝ a partition P such that

$U(P f,g) - L(P,f,g) < \varepsilon$ **------(1)** break partition $P = \{x_0,x_1 --- x_{r-1} ,x_r =c ----- x_{n-1},x_n\}$ such that $x_r = c$ then $U(P,f,g) = \sum^n_{i=1} M_i\ \delta g_i = \sum^r_{i=1} M_i\ \delta g_i + \sum^n_{i=r+1} M_i\ \delta g_i$ **----------(2)**

and $L(P,f,g) = \sum^n_{i=1} m_i\ \delta g_i = \sum^r_{i=1} m_i\ \delta g_i + \sum^n_{i=r+1} m_i\ \delta g_i$ **------------------------------(3)**

Subtracting (2) &(3) and using (1) we have $\sum^r_{i=1} (M_i - mi)\delta gi + \sum^n_{i=r+1}(M_i - m_i)\delta gi < \varepsilon$

i.e. $\sum^r_{i=1} (M_i - mi)\delta gi < \varepsilon$ and $\sum^n_{i=r+1} (M_i - m_i)\delta gi < \varepsilon$ -------------------(4) ,also $f \in R\ (g)$

$\Rightarrow$ f is bounded in[a, b] $\Rightarrow$ f is bounded in [a,c] & [c,b] both -- ----(5) therefore

$f \in R(g)$on [a,c] and $f \in R(g)$ on [c, b] by (4) & (5). Again we have

$\sum^{n}_{i=1} M_i\ \delta g_i = \sum^{r}_{i=} M_i\ \delta g_i + \sum^{n}_{i=r+1} M_i\ \delta g_i$ whence making $||P|| \rightarrow 0$, by definition of R-S sums we get $\int^{b}_{a} fdg$

$= \int^{c}_{a} fdg + \int_{c}^{b} fdg$, hence the result.

LECTURE – 4

Today we shall discuss some more theorems on R-S Integral

Theorem13 : If $f \in R(g_1)$ and $f \in R(g_2)$ then $f \in R(g_1 + g_2)$ and $\int_a^b fd(g_1 + g_2)$

$= \int_a^b fdg_1 + \int_a^b f\, dg_2\, fdg_2$.

Proof : Since $f \in R(g_1)$ so there exists a partition P_1 such that

$U(P_1,f,g_1) - L(P_1,f,g_1) < \varepsilon$ ---------------------- **(1)** and as $f \in R(g_2)$ so there exists a partition P_2 such that $U(P_2\, f,g_2) - L(P_2,f,g_2) < \varepsilon$ ---------------**(2)**

Let P be the common refinement of P_1 & P_2 i.e. $P = P_1 \cup P_2$ then from (1) & (2) we have $U(P, f,g_1) - L(P,f,g_1) < \varepsilon$ ----------- (**3**) & $U(P, f,g_2) - L(P,,f,g_2) < \varepsilon$ ------------**(4)**

Let $g = g_1+g_2$ then consider $\sum_{i=1}^n M_i\, [g(x_i)-g(x_{i-1})] = \sum_{i=1}^n M_i\, [(g_1+g_2)(x_i) - (g_1+g_2)(x_{i-1})]$ $= \sum_{i=1}^n M_i[(g_1+g_2)(x_i)\,(g_1+g_2)(x_{i-1})] = \sum_{i=1}^n M_i\, [(g_1(x_i)- g_1(x_{i-1})] + \sum_{i=1}^n M_i[(g_2(x_i)-g_2(x_{i-1})]$.

Thus $U[P,f,g,] = U[P,f,g_1]+ U[P,f,g_2]$ ----------- **(5).** Similarly it can be proved that

$L[P,f,g] = L(P,f,g_1] + L(P,f,g_2]$ -------------------------**(6)** so, $U[P, f, g,]- L[P, f,g] =$

$U[P,f,g_1]- L(P,f,g_1] + U[P,f,g_2] -L(P,f,g_2] < \varepsilon$,hence $f \in R(g_1+g_2)$ as $g = g_1+g_2$ By (5) we get $\inf.U[P,f,g] \geq \inf.U[P,f,g_1] + \inf.U[P,f,g_2]$ so $\int_a^b f\, dg \geq \int_a^b fdg_1 + \int_a^b f\, dg_2$----**(7)**

and $U[P,f,g,] = U[P,f,g_1]+ U[P,f,g_2] \Rightarrow \int_a^b fdg \leq \int_a^b fdg_1 + \int_a^b f\, dg_2$ --------------------**(8)**

by (7) & (8) we have $\int_a^b fd(g_1+g_2) = \int_a^b fdg_1 + \int_a^b f\, dg_2\, fdg_2$.

Theorem14 : If $f \in R(g)$ on $[a,b]$ and c is a positive constant then $f \in R(cg)$ and

$\int_a^b f\, d(cg) = c\int_a^b fdg$

Proof : As $f \in R(g)$on [a, b),so for given $\varepsilon > 0$ there exists partition P such that

$U[P, f, g,] - L[P, f, g]<\varepsilon/c \Rightarrow \sum^{n}_{i=1} M_i [(cg(x_{i)}) - cg(x_{i-1})] - \sum^{n}_{i=1} m_i [(cg(x_i\ ) - cg(x_{i-1})] < \varepsilon$

$\Rightarrow U[P, f, cg]–L[P, f, cg] < \varepsilon$, so $f \in R(cg)$ on [a, b]. Also as $L[P ,f, cg] = \sum^{n}_{i=1} m_i\, c\, \delta gi$

$= \sum^{n}_{i=1} m_i [(c\{g(x_i\ ) - g(x_{i-1})\}] = c \sum^{n}_{i=1} m_i\, \delta gi = c\, L[P,f, g]$. As $\int^{b}_{a} fd(cg) = \text{Sup } L[P,f, cg]$

$= c \text{ Sup } L[P,f, g]= c \int^{b}_{a} fdg$. Hence $\int^{b}_{a} fd(cg) = \int^{b}_{a} fdg$ is proved

Theorem15:If $f\in R(g)$ on [a,b]and if $|f (x)|\le M$ on[a,b],then$| \int^{b}_{a} fdg|\le M [g(b) - g(a)]$

Proof : Since $|f(x)| \le M$ i.e. $–M \le f(x) \le M$ we have

$M [g (b)–g(a)] \le |\int_a^b f(x)\, dg| \le M [g (b) - g(a)]$, by Mean value Theorem we have $|\int_a^b f(x)\, dg| = M[g (b) - g(a)]$. Hence proved

LECTURE -5

Today we will discuss the theorem which states the relation betwwen RIEMANN & R-S Integral (Th.16),First Mean value Theorem and some other T*heorems* ***&cor.***

Theorem16 : Assume g is monotonically increasing and $g' \in R$ on [a,b].Let f be bounded real function on[a ,b] then $f \in R(g)$ if and only if $f g' \in R(g)$,in that case

$\int_a^b f dg = \int_a^b f(x)g'(x)dg$ ----------- **(1)**

Proof : Let $\varepsilon > 0$ be given, since $g' \in R$ then by Th. 3 to g' , there is a partition

$P = \{ a = x_0, x_1, x_2 \text{-------} x_{n-1}, x_n = b \}$ of [a ,b] such that $U[P, g'] - L[P, g'] < \varepsilon$.**----------(2).**

The Mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$,such $\delta g_i = g'(t_i)\ \delta x_i$,

for i=1,2,---n. If $s_i \in [x_{i-1}, x_i]$ then $\sum_{i=1}^{n} |g'(s_i) - g'(t_i)|\ \delta x_i < \varepsilon$ ----------------------- **(3)**

By (2) & Th. 4. Now Put $M = \text{Sup}\ |f(x)|$,since $\sum_{i=1}^{n} f(s_i)\ \delta g_i = \sum_{i=1}^{n} f(s_i)g'(t_i)\ \delta x_i$,it follows from (3) that $|\sum_{i=1}^{n} f(s_i)\ \delta g_i - \sum_{i=1}^{n} f(s_i)g'(t_i)\ \delta x_i| \le M\varepsilon$. **------------------(4).**

In particular $\sum_{i=1}^{n} f(s_i)\ \delta g_i \le U[P, f] + M\varepsilon$, for all choices of $s_i \in [x_{i-1}, x_i]$, so that

$U[P, f, g] \le U[P, f, g'] + M\varepsilon$, the same argument leads from (4) to

$U[P,f g'] \le U[P,f g] + M\varepsilon$ .Thus $|U[P,f,g] - U[P,fg']| \le M\varepsilon$. **------------------------ (5)**

Now (2) is true if P is replaced by any refinement, hence (5) also remains true .We conclude that $|\overline{\int}_a^b f(x)\, dg - \overline{\int}_a^b f(x)g'(x)\, dx| \le M\varepsilon$. But ε is arbitrary ,hence

$\overline{\int}_a^b$ **f(x) dg =** $\overline{\int}_a^b$ **f(x)g'(x)dx** for any bounded function f. The equality of the lower integrals follows from (4) in the same manner. Hence the theorem is proved.

Theorem 17 : Let f ϵR (g) on [a, b] then m [g(b) – g(a)] ≤ $\int_a^b$ f(x) dg ≤ M [g (b)- g (a)]

Proof : We have m ≤ $m_r \le M_r$ ≤ M therefore $\sum_{r=1}^{n} m\, \delta g_r \le \sum_{r=1}^{n} m_r\, \delta g_r \le \sum_{r=1}^{n} M_r\, \delta g_r \le \sum_{r=1}^{n} M\, \delta g_r$ i.e .[g(b) - g(a)] ≤ L(P,f) ≤ U(P,f) ≤ M [g (b) – g(a)].But L[P,f,g]≤ $\overline{\int}_a^b$ f(x) dg ≤ $\underline{\int}_a^b$ f(x) ≤ M [g (b) - g(a)]. Since f ϵ R (g) hence $\overline{\int}_a^b$ f(x) dg = $\underline{\int}_a^b$ f(x) = $\int_a^b$ f(x), it follows that m[g(b) – g(a)] ≤ $\int_a^b$ f(x) dg ≤ M [g (b) - g(a)]. Hence proved.

Cor1. If f ϵ R (g) then Ǝ a number ζ lying between m & M such that

$$\int_a^b f(x)\, dg = \zeta\, [g(b) - g(a)] \quad \text{---------------------------------- -(1)}$$

Proof: As we have proved m [g (b)– g(a)] ≤ $\int_a^b$ f(x) dg ≤ M [g (b)–g(a**)].--------(2)**

Then Ǝ a number ζ such that m ≤ ζ ≤ M it follows that $\int_a^b$ f(x) dg = ζ [g (b) – g(a)]

Cor2. First Mean value theorem : If f is continuous and real , g is monotonically increasing on [a, b] ,then Ǝ a point c ϵ(a, b)such that$\int_a^b$ f(x)dg = f(c)[g (b)- g(a)]

Proof : From above cor.1,we have $\int_a^b$ f(x) dg = ζ[g (b)- g(a)],f is continuous in [a, b] and m ≤ ζ ≤ M ⇒ Ǝ c ϵ [a, b] such that f(c) = ζ **-----------(3)** ,by (1) & (3) we have $\int_a^b$ f(x)dg = f(c)[g (b) - g(a)]

Cor. 3: If f ϵ R (g) and if |f(x)| ≤ k on [a, b] then |$\int_a^b$ f(x) dg| = k [g (b) – g(a)]

Proof : Since |f(x)| ≤ k ie. –k ≤ f(x) ≤ k, we have

$-k\ [g(b) - g(a)] \le |\int_a^b f(x)\, dg| \le k\ [g(b) - g(a)]$, by Mean value Theorem we have

$|\int_a^b f(x)\, dg| = k\ [g(b) - g(a)]$. Hence proved

Integration and Differential Theorem 18 : Let $f \in R$ on $[a, b]$ .For $a \le x \le b$,put

$F(x) = \int_a^b f(t)\, dt$,then F is continuous on [a, b] ,furthermore, if f is continuous at a point x_0 of [a, b] ,then F is differential at x_0 , and $F'(x_0) = f(x_0)$.

Proof : Since $f \in R$,f is bounded . Suppose $|f(t)| \le M$,for $a \le t \le b$. If $a \le x \le b$,then

$|F(y) - F(x)| = |\int_x^y f(t)\, dt| \le M(y - x)$ .By Th. 14 & 15 given ε, we see that

$|F(y) - F(x)| < \varepsilon$ provided that $|y - x| < \varepsilon/M$ .this proves continuity & in fact uniform continuity of F. Now suppose f is continuous at x_0 ,hence given $\varepsilon > 0$,choose $\delta > 0$ such that $|F(t) - F(x_0)| < \varepsilon$ if $|t - x_0| < \delta$ and $a \le t \le b$ .hence if

$x_0 - \delta \le s \le x_0 \le t \le x_0 + \delta$ and $a \le s \le t \le b$, we have by Th.15

$\{F(t) - F(s)/(t-s)\} - f(x_0)|$

$= |1/(t-s) \int_a^b \{f(u) - f(x_0)\}\, dg| < \varepsilon$ it follows that $F'(x_0) = f(x_0)$.

LECTURE -6

Now we will study about Fundamental Theorem of calculus ,integration by parts and some problems on R-S Integral

Theorem19 : The Fundamental Theorem of Calculus: If $f \in R$ on [a, b] and if there is a differential function F on [a, b] such that $F' = f$,then $\int_a^b f(x)\,dx = F(b) - F(a)$

Proof : **Let ε >0 be given , choose a partition** $P = \{ a= x_0, x_1, x_2 \text{-------} x_{n-1}, x_n = b \}$ of [a, b] so that $U[P, f] - L[P, f] < \varepsilon$,the mean value theorem furnishes points

$t_i \in [x_{i-1}, x_i]$ such that $F(x_i) - F(x_{i-1}) = f(t_i)\,\delta x_i$, for i =1,2,--------n thus

$\sum_{i=1}^{n} f(t_i)\,\delta x_i = F(b) - F(a)$,it follows from Th. 4 that $|F(b) - F(a) - \int_a^b f(x)\,dx| < \varepsilon$,this holds for every $\varepsilon > 0$ hence the proof is complete.

Theorem20: Suppose F & G are differential function on [a ,b] ,$F'=f \in R$ & $G'=g \in R$ then $\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$.

Proof : Put $H(x) = F(x)G(x)$ and apply Th.17 to H and its derivative, from this We have $\int_a^b H'(x)\,dx = H(b) - H(a)$ **-----------(1)** [because $\int_a^b F'(x)\,dx = F(b) - F(a)$].

Now $H'(x) = F'(x)G(x) + F(x)G'(x) = f(x)G(x) + g(x)F(x)$ then from (1) we have

$\int_a^b H'(x)\,dx = \int_a^b \{ f(x)G(x) + g(x)F(x)\}\,dx = H(b) - H(a) = F(b)G(b) - F(a)G(a)$

$= \int_a^b f(x)G(x)\,dx + \int_a^b g(x)F(x)\,dx$, thus we have

$\int_a^b F(x)g(x)dx = F(b)G(b) - Fa)G(a) - \int_a^b f(x)G(x)dx$.Hence proof is complete

Now We solve Some problems on R-SIntegral :

Prob.1 : Evaluate RS $\int_0^1 x dx^2$

Sol. Since x is continuous and x^2 is increasing in [0,1] RS $\int_0^1 x\, dx^2$ exists .to find its value ,consider the partition P = { x_0 =0,x_1,x_2-------x_{n-1},x_n =1 }where x_r = r/n let

$\zeta_r \in [x_{r-1},x_r\}$ then $\sum_{r=1}^n f(\zeta_r)\,\delta g_r = \sum_{r=1}^n x_r(x^2_r - x^2_{r-1}) = \sum_{r=1}^n\{(r/n)(r/n)^2 - (r-1/n)^2\} =$

$1/n^3 \sum_{r=1}^n r(2r-1) = 1/n^3[2\sum_{r=1}^n r^2 - \sum_{r=1}^n r] = (n+1)(4n-1)/6n^2 = \text{Lim}_{n\to\infty}(n+1)(4n-11)/6n^2$

= (1) (4) /6 = 2/3

Another method : Since f (x) = x is R - Integrable and g(x) = x^2 is differentiable on [0,1] then by Th.16, we have R $\int_0^1 x dx^2 = \int_0^1 x2x = 2\int_0^1 x^2 dx = 2/3$

Prob.2 : Evaluate the following (1) $\int_0^2 x^2 dx^2$ (2) $\int_0^1 x^2 dx^2$

Sol. Here we use the result $\int_a^b fdg = \int_a^b fg'dx$

(1) $\int_0^2 x^2 dx^2 = \int_0^2 x^2(2x)\, dx = 2\int_0^2 x^3\, dx = 2/4\,(2^4) = 8$

(2) $\int_0^1 x^2 dx^2 = \int_0^1 x^2(2X)\, dx = \int_0^1 x^3 dx = 2/4\,(1)^4 = 1/2$

Prob3: Find the value of $\int_{-1}^2 x^3\, d\,|x|^5$

Sol: We have $\int_{-1}^2 x^3\, d\,|x|^5 = \int_{-1}^0 x^3\, d\,(-x)^5 + \int_0^2 x^3\, d\,(x^5) = -5\int_{-1}^0 x^7\, d\,x + 5\int_0^2 x^7\, d\,x =$ 5/8 +160

Prob.4: Evaluate $\int_0^2 [\,x]\, dx^2$

Sol: $\int_0^2 [\,x]\, dx^2 = \int_0^2 [\,x]\, 2x\, dx = 2\int_0^1 [\,x\,]x\, dx + 2\int_1^2 [x]x\, dx = 2\int_0^1 0\, x\, dx + 2\int_1^2 1.x\, dx =$ $0 + (x^2)_1^2 = 4 - 1 = 3$