02.02.23

Lecture - 2

Question - 1 Find the first order partial derivatives of $u = \cos^{-1}\left(\frac{x}{y}\right)$.

$$\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left[\frac{1}{y}\right].$$

$$= \frac{-y}{\sqrt{y^2-x^2}} \times \frac{1}{y}$$

$$= \frac{-1}{\sqrt{y^2-x^2}}. \quad \text{Ans.}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left[\frac{-x}{y^2}\right]$$

$$= \frac{x}{y\sqrt{y^2-x^2}}. \quad \text{Ans.}$$

Question-2 If $z(x+y) = x^2+y^2$, Show that :-

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

$$z = \frac{x^2+y^2}{x+y}.$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2+y^2)}{(x+y)^2}$$

$$= \frac{x^2+2xy-y^2}{(x+y)^2}.$$

Since the function is symmetric,

$$\frac{\partial z}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2}.$$

Now,

LHS:- $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{x^2+2xy-y^2 - y^2-2xy+x^2}{(x+y)^2}\right)^2$

$$= \left(\frac{2x^2 - 2y^2}{(x+y)^2}\right)^2$$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

RHS:- $4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

$$= 4\left(1 - \frac{x^2 - 2xy + y^2}{(x+y)^2} - \frac{y^2 - 2xy + x^2}{(x+y)^2}\right)$$

$$= 4\left(\frac{(x-y)^2}{(x+y)^2}\right).$$

$\because$ LHS = RHS

Hence proved. Ans.

Question-3 If $z = e^{ax+by} f(ax - by)$. Show that

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)\cdot a + f(ax-by)\, e^{ax+by}$$

$$b\frac{\partial z}{\partial x} = ab\, e^{ax+by}\left(f'(ax-by) + f(ax-by)\right) \quad \text{------ } 1.$$

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)\cdot(-b) + f(ax-by)\, e^{ax+by}\cdot b$$

$$a\frac{\partial z}{\partial y} = ab\, e^{ax+by}\left(f(ax-by) - f'(ax-by)\right) \quad \text{------ } 2.$$

now, on adding 1. and 2.,

LHS:- $\frac{b\,\partial z}{\partial x} - \frac{a\,\partial z}{\partial y}$

$$= ab e^{ax+by}\left(f'(ax-by) + f(ax-by) - f'(ax-by) + f(ax-by)\right)$$

$$= ab e^{ax+by}\left(2f(ax-by)\right)$$

$$= 2ab\left(e^{ax+by} f(ax+by)\right)$$

$= 2abz.$:- RHS.

Hence Proved. Ans.

Question-4. If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$.

Show that :- $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 x.$$

Now,

$$\sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x}{(\tan x + \tan y + \tan z) \cdot \cos^2 x}$$

$$= \frac{2 \sin x}{(\tan x + \tan y + \tan z) \cos x}$$

$$= \frac{2 \tan x}{(\tan x + \tan y + \tan z)} \quad \text{------ } 1.$$

Similarly,

$$\sin 2y \frac{\partial u}{\partial y} = \frac{\sin 2y}{(\tan x + \tan y + \tan z) \cos^2 y}.$$

$$= \frac{2 \tan y}{(\tan x + \tan y + \tan z)} \quad \text{------ } 2.$$

Similarly,

$$\sin 2z \frac{\partial u}{\partial z} = \frac{2 \tan z}{(\tan x + \tan y + \tan z)} \quad \text{------ } 3.$$

On adding 1., 2. and 3.,

LHS :- $2\left(\frac{\tan x + \tan y + \tan z}{\tan x + \tan y + \tan z}\right)$

$= 2.$

:- RHS

Hence Proved. Ans.