03.02.23

Lecture - 3

Question - 1 If $u = f(r)$, where $r^2 = x^2 + y^2$, prove that :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

$$\frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x}$$

Now,

$$r^2 = x^2 + y^2.$$

$$2r \frac{\partial r}{\partial x} = 2x.$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

$$\boxed{\frac{\partial u}{\partial x} = f'(r) \frac{x}{r}.}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = f'(r) \frac{1}{r} + f'(r) x \left(\frac{-1}{r^2}\right) \cdot \frac{\partial r}{\partial x} + \frac{x}{r} f''(r) \frac{\partial r}{\partial x}$$

$$= f'(r) \cdot \frac{1}{r} + f'(r) \left(\frac{-x^2}{r^3}\right) + \frac{x^2}{r^2} f''(r). \quad \text{------ } 1.$$

$\because$ the function is symmetrical,

Hence,

$$\frac{\partial^2 u}{\partial y^2} = f'(r) \cdot \frac{1}{r} - f'(r) \frac{y^2}{r^3} + \frac{y^2}{r^2} f''(r) \quad \text{------ } 2.$$

Now,

LHS :- $2 f'(r) \cdot \frac{1}{r} - f'(r) \frac{x^2}{r^3} - f'(r) \frac{y^2}{r^3} +$

$$f''(r) \frac{x^2}{r^2} + f''(r) \frac{y^2}{r^2}.$$

$$= 2 f'(r) \frac{1}{r} - \frac{f'(r)}{r^3}(x^2 + y^2) + f''\frac{(r)}{r^2}(x^2 + y^2)$$

$$= 2 f'(r) \cdot \frac{1}{r} - f'(r) \frac{1}{r} + f''(r).$$

$$= f'(r) \cdot \frac{1}{r} + f''(r) \quad \text{:- RHS.}$$

Hence Proved. Ans.

Assignment questions

1. Find the first and second order partial derivatives of $u = \log(x^2 + y^2)$.

2. If $u = x^2 + y^2 + z^2$, show that $x u_x + y u_y + z u_z = 2u$.

3. If $z = \log(x^2 + xy + y^2)$, show that $x u_x + y u_y = 2$.

4. If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, show that $f_x + f_y + f_z = 0$.

5. If $z = f(x + ct) + \phi(x - ct)$, show that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

———x———x———