08.03.23

Lecture-4

Homogeneous Function :- A function $f(x, y)$ is said to be homogeneous of degree n in variables x and y if it can be expressed in the form of $x^n \phi(y/x)$ OR $y^n \phi(x/y)$.

Alternative Test for a function $f(x, y)$ to be homogeneous of degree n :- is that if $f(tx, ty) = t^n f(x, y)$ then it is a homogeneous function.

Example :- if $f(x, y) = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$

$$= \frac{x(1 + y/x)}{\sqrt{x}(1 + \sqrt{y/x})}$$

$$= x^{1/2} \phi(y/x).$$

Hence, the given function is homogeneous with degree $n = 1/2$. Ans. in variable x and y.

Method 2 :-

$$= \frac{x + y}{\sqrt{x} + \sqrt{y}}$$

$$= \frac{y(x/y + 1)}{\sqrt{y}(\sqrt{x/y} + 1)}$$

$$= y^{1/2} \phi(x/y).$$

Hence, the function is homogeneous. Ans.

Method 3:-

$$= \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}}$$

$$= \frac{t(x+y)}{\sqrt{t}(\sqrt{x} + \sqrt{y})}$$

$$= t^{1/2} f(x, y).$$

Hence, the function is homogeneous Ans.

Similary, a function $f(x, y, z)$ is said to be homogeneous of degree n in x, y and z. if it can be expressed as

$$f(x, y, z) = x^n \phi(y/x, z/x)$$

OR

$$= y^n \phi(x/y, z/y)$$

OR

$$= z^n \phi(x/z, y/z).$$

Alternative test:- is that if $f(tx, ty, tz) = t^n f(x, y, z)$ then it is a homogeneous function.

Euler's Theorem for homogeneous function:-
If u is a homogeneous function of x and y of degree n then $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$.

Proof:-
$\because$ u is a homogeneous function of degree n in x and y, we have,

$$u = x^n \phi(y/x) \qquad \{\text{By Definition}\}.$$

$$\frac{\partial u}{\partial x} = x^n \phi'(y/x)\left(\frac{-y}{x^2}\right) + \phi(y/x) \cdot n x^{n-1}$$

Now,

$$x \frac{\partial u}{\partial x} = -yx^{n-1} \phi'(y|x) + \phi(y|x) \cdot nx^n \quad \ldots\ldots 1.$$

$$\frac{\partial u}{\partial y} = x^n \cdot \phi'(y|x) \cdot \frac{1}{x}$$

$$y \frac{\partial u}{\partial y} = yx^{n-1} \phi'(y|x) \quad \ldots\ldots 2.$$

on adding 1. and 2., we get,

LHS:-

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

$= nx^n \phi(y|x)$

$= nU.$

$:-$ RHS.

Hence proved. Ans.

Theorem:- If u is a homogeneous function of degree n in x and y, then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Proof:-

$\because$ u is a homogeneous of degree n in x and y

then by Euler's Theorem we have,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu. \quad \ldots\ldots 1.$$

Differentiating eq 1., partially with respect to we have,

$$\Rightarrow x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{n \partial u}{\partial x} \quad \ldots\ldots 2.$$

Now, differentiating eq. 1, partially with respect to $y \rightarrow$

$$= x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{n \partial u}{\partial y}$$