07.02.23

Lecture-5

But, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Now,

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} \quad \text{.........} \quad 3.$$

Multiplying 2. by x and 3. by y. we get, and adding them,

$$= x^2\frac{\partial^2 x}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(y\frac{\partial u}{\partial y} + x\frac{\partial u}{\partial x}\right) = n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n^2u - nu.$$

$$= x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Hence Proved. Ans.

Theorem:- If $F(u) = V(x, y)$, where V is a homogeneous function in x and y of degree n then

i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nF(u)}{F'(u)}$.

ii) $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = g(u)[g'(u) - 1]$.

where $g(u) = \frac{nF(u)}{F'(u)}$.

Proof:-

i) $\because$ $F(u) = V(x, y)$ is a homogeneous function of degree n, then by Euler's theorem we get. $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = nV$.

$$= x\frac{\partial F(u)}{\partial x} + y\frac{\partial F(u)}{\partial y} = nF(u).$$

$$= xF'(u)\frac{\partial u}{\partial x} + yF'(u)\frac{\partial u}{\partial y} = nF(u).$$

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)} \quad \text{------- } 1.$$

Hence Proved. Ans.

ii) Let $\phi(u) = \frac{nF(u)}{F'(u)}$, then from 1,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \phi(u). \quad \text{------- } 2.$$

Differentiating eq. 2, partially with respect to x we have,

$$= x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \phi'(u)\frac{\partial u}{\partial x}.$$

Now, Differentiating eq. 2, partially with respect to y. we have,

$$= x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \phi'(u)\frac{\partial u}{\partial y}$$

Now,

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (\phi'(u) - 1)\frac{\partial u}{\partial x}. \quad \text{------- } 3.$$

and

$$y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial x \partial y} = (\phi'(u) - 1)\frac{\partial u}{\partial y} \quad \text{------- } 4.$$

Now, multiplying eq. 3 by x and eq. 4 by y, we get

$$= x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} =$$

$$(\phi'(u) - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [\phi'(u) - 1]\,\phi(u)$$

Hence Proved. Ans.

question-1 verify Euler's theorem for the function $u = (x^{1/2} + y^{1/2})(x^n + y^n)$.

$$u = x^{1/2}\left(1 + (y/x)^{1/2}\right) x^n \left(1 + (y/x)^n\right)$$

$$= x^{n+1/2}\,\phi(y/x)$$

. This implies that u is a homogeneous function of degree $n + 1/2$ in variables x. and y.

∴ By Euler's theorem, we have to show that

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)u.$$

Now,

$$u = (x^{1/2} + y^{1/2})(x^n + y^n).$$

$$\frac{\partial u}{\partial x} = (x^{1/2} + y^{1/2})\,nx^{n-1} + (x^n + y^n)\frac{1}{2}x^{-1/2}. \quad \text{--- } 1$$

Similarly,

$$\frac{\partial u}{\partial y} = (x^{1/2} + y^{1/2})\,ny^{n-1} + (x^n + y^n)\frac{1}{2}y^{-1/2} \quad \text{--- } 2.$$

Now, multiplying eq. 1 by x and eq. 2 by y we have,

$$x\frac{\partial u}{\partial x} = (x^{1/2} + y^{1/2})\,nx^n + (x^n + y^n)\frac{1}{2}x^{1/2}. \quad \text{--- } 3.$$

$$y\frac{\partial u}{\partial y} = (x^{1/2} + y^{1/2})\,ny^n + (x^n + y^n)\frac{1}{2}y^{1/2}. \quad \text{--- } 4.$$

Now, on adding 3. and 4.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)(x^n + y^n)(x^{1/2} + y^{1/2})$$

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)u.$$

Hence Proved. Ans.